THE STABLE LÜROTH PROBLEM: A SURVEY ON RECENT TECHNIQUES AND EXAMPLES.

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Abstract. We survey the recent specialisation method by Voisin and, by means of Colliot-Thélène and Pirutka's generalisation, we show many notable applications in the stable Lüroth problem. Furthermore, we analyse the current results in the stable Lüroth problem for conic fibrations, also attempting to present some possible new questions.

1. Introduction

One of the most fascinating areas of algebraic geometry is the birational classification of varieties, namely grouping together algebraic varieties in equivalence classes up to birational transformations. This has some significant limitations. For instance, one could start subdividing the class of algebraic varieties in *rational* and *irrational* ones. Even if this were possible in general, the amount of different behaviours in the class of irrational varieties is so huge to make this simple classification almost useless. The same problem, namely the excessive vastity of birational models, appears studying classification up to birational equivalence.

For these reasons, a finer classification is needed and some notions of "nearly rational" varieties have been introduced since the late XIX century. We report here the basic definitions.

Definition 1.1. A projective variety X defined over a field k is called

1. *rational* if there is a birational map $X \dashrightarrow \mathbf{P}^n_k$ for some $n > 0$.
2. *stably rational* if $X \times \mathbf{P}^m_k$ is rational;
3. *unirational* if there is a dominant rational map $\mathbf{P}^m_k \dashrightarrow X$;
4. *rationally connected* if, for every algebraically closed field extension L/k and for each $p, q \in X(L)$, namely there is a morphism $\mathbf{P}^1_L \longrightarrow X$ such that $0 \mapsto p$ and $\infty \mapsto q$.

There are some easily seen implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4), which however do not admit reverse statements in general. To distinguish the two classes of rational and unirational varieties has been one of the most challenging problems in algebraic geometry until the second half of XX century, and still remains today a very fertile research field. Moreover, the notions depends heavily on the ground field, with the classical case (k algebraically closed with $\operatorname{char} k = 0$) somehow better understood than the positive characteristic one.

However, all the definitions above are equivalent in low dimension, at least over the field $k = \mathbf{C}$ of complex numbers. We have, indeed, the following result.

Theorem 1.2. (Lüroth) *Let k be a field such that $\operatorname{char} k \neq 2$. Then every subfield of $k(t)$ is a purely transcendetal field extension of k.*

In geometric terms, the Lüroth Theorem proves that every unirational curve is rational. The general version, which aims to decide whether or not unirational and rational varieties are equivalent, takes the name of *generalised Lüroth problem*, or *generalised stable Lüroth problem* if the attention is restricted to differences between unirational and stably rational varieties. A positive answer is also known for projective surfaces, thanks to the Castelnuovo-Enriques characterisation.

Theorem 1.3. (Castelnuovo) *Let X be a projective surface over an algebraically closed field k such that $\operatorname{char} k = 0$. Therefore X is rational if and only if $\dim H^1(X, \mathscr{O}_X) = 0 = \dim H^0(X.\omega_X^{\otimes 2})$.*

In fact, if X is unirational, then there is a dominant map $\mathbf{P}^2_k \dashrightarrow X$. Blowing up $\mathbf{P}^2_k$ repeatedly, we can extend this map to a morphism $\mathrm{Blow}(\mathbf{P}^2_k) \longrightarrow X$ and then we can bound $\dim H^1(X, \mathscr{O}_X)$ and $\dim H^0(X, \omega_X^{\otimes 2})$ by means of the corresponding quantities in $\mathrm{Blow}(\mathbf{P}^2_k)$. But the blow-up of the projective plane is rational, hence by Castelnuovo's Theorem, X is rational too.

It was an open question to decide whether or not the same result held in dimension at least 3, and in general the Lüroth problems become non-trivial in higher dimension. The first examples of unirational non-rational 3-folds were discovered during the Seventies by Emile Artin and David Mumford. Subsequently, more examples have been provided; here we shall mention some notable ones.

- Smooth cubic 3-folds in $\mathbf{P}^4_{\mathbf{C}}$ are all unirational, but they can't be rational. This has been proved by Clemens and Griffiths using the method of intermediate jacobians. Stable rationality of this class of varieties is still unknown; the class of cubics seems currently the most difficult to approach for this kind of problems.
- Artin and Mumford constructed an unirational variety which is not stably rational (and *a fortiori ratione*, not rational): it is a quartic, 3-dimensional variety X which is a double cover $X \longrightarrow \mathbf{P}^3_{\mathbf{C}}$ ramified along a special cubic surface with 10 nodes. The example is illustrated further in details.
- The very general quartic double solid is unirational but not stably rational. This result is due to Claire Voisin and comes from an application of the specialisation method.
- It is not known if every quartic 3-fold in $\mathbf{P}^4_{\mathbf{C}}$ is unirational, but Alena Pirtuka and Jean-Louis Colliot-Thélène proved that the very general member of this family is not stably rational (see further for details).

- Conic fibrations, and more generally, quadric fibrations are a particular class of 3-folds, some of which are unirational. Imposing some reducibility conditions on the discriminant locus, it can be proved that they are not stably rational. Some details and techniques are discussed further.

Stable irrationality is a subtle condition to detect. Indeed, most of the birational invariants that have been used in various contexts are trivial or very difficult to compute in many examples. To address the difficulties, the idea is to study *degenerations* of the variety X in which we are interested, namely to realise X as the special fibre $\mathfrak{X}_0$ of a suitable flat morphism

$$\mathfrak{X} \longrightarrow B$$

where B is a smooth curve. Assume that the generic fibre $\mathfrak{X}_t$ has a particular property $\mathbb{P}$, which behaves well under specialisation; then if X is not "too singular", the property $\mathbb{P}$ extends to each smooth model $\widetilde{X} \simeq_{\mathrm{bir}} X$.

Of course, the property $\mathbb{P}$ has to be wisely chosen in a way that $\mathbb{P}$ can be both "transferred" from general fibres to the special fibre, and have some strong connection to stable rationality. One could expect $\mathbb{P}$ to be non-triviality of some stable rational invariant, but this is in general not sufficient. For instance, in the Artin-Mumford example, stable irrationality of X is proved by showing non-triviality of a cohomology class in $\mathrm{Br}(X) = \operatorname{tors} H^3(X(\mathbf{C}), \mathbf{Z})$. However, this group is not useful to detect stable irrationality of varieties by specialisation, as it vanishes for a general quartic double solid.

For these reasons, new invariants have been introduced in the recent years, constructed with the specific purpose to be exploited via specialisation. These invariants, which are essentially of Chow-theoretic nature, have the disadvantage to lack a immediate geometric meaning, but they appear to be particularly suitable for this kind of obstruction problems.

2. Decomposition of the diagonal and zero-cycles

In this section we introduce the key invariants for the study of stable rationality. In this setting, we assume k to be a general field (possibly, of characteristic 2, even though we will soon restrict to simpler cases).

Let X be a projective variety of dimension n over a field k. Recall that the *diagonal* of X is the variety

$$\Delta_X := \{(x, x) \mid x \in X\} \subseteq X \times X$$

Suppose also that $X(k) \neq \varnothing$ and let us give the following definition.

Definition 2.1. We say that X has *Chow decomposition of the diagonal* if

$$[\Delta_X] = [X \times x] + [Z] \in \mathrm{CH}_n(X \times X)$$

where $x \in X(k)$ and Z is a n-cycle on $X \times X$ supported on some $D \times X$, where $D \subset X$ is a closed proper subvariety (in the sense that $D \neq X$).

If X has nice properties, then the Chow decomposition of the diagonal does not depend on the choice of x, which can be replaced with a zero-cycle. Assuming X proper is enough to guarantee this.

Definition 2.2. We say that a projective variety X over k has *universally trivial* CH_0 *group* (briefly, X is *universally* CH_0 *trivial*) if, for any field extension L/k, we have $\mathrm{CH}_0(X_L) \simeq \mathbf{Z}$ via the degree map, where $X_L := X \times_k \mathrm{Spec}(L)$.

If X is a projective integral variety over k and X is universally CH_0 trivial, then X admits a Chow decomposition of the diagonal ([7]). If in addition X is smooth, the two conditions are equivalent.

Theorem 2.3. *Let X be a smooth projective variety over k and assume that X has a Chow decomposition of the diagonal. Then X is universally* CH_0 *trivial.*

The point of view of Chow decomposition of the diagonal is more geometric and can be rephrased in cohomological fashion, giving rise to a notion which is simpler to check but almost equally significant in many cases (see [19]).

Now, the main motivation to study the above properties is that they are stable birational invariants.

Theorem 2.4. *Admitting a Chow decomposition of the diagonal is a stable birational invariant for smooth projective varieties. In this setting, being universally* CH_0*-trivial is also a stable birational invariant.*

Since the projective space $\mathbf{P}^n_k$ admits a Chow decomposition of the diagonal, stably rational varieties admit a Chow decomposition of the diagonal. Of course the existence of a decomposition is not a sufficient condition for stable rationality (see [18]) but has equally been applied to explore generalised invariants for stable rationality; a survey of the most important stable birational invariants can be found in [19].

Our view point will be the *non existence*: if X is a smooth projective variety over an infinite algebraically closed field k, with $\operatorname{char} k = 0$, such that it does not admit a Chow decomposition of the diagonal, then X can not be stably rational.

3. Specialisation method

The two invariants defined in the above section, together with the non-existence point of view, can be successfully employed to study varieties that come in *families* with some properties. In this respect we state and prove here the following *specialisation principle.* From now, we assume that the varieties are defined over a field k which is algebraically closed with $\operatorname{char} k = 0$.

Theorem 3.1. *Let $f : \mathfrak{X} \longrightarrow B$ be a proper, flat projective morphism of relative dimension at least 2, where B is a smooth curve. Assume that the* *fibres $\mathfrak{X}_t$ are smooth for $t \neq 0$ and that it has at worst quadratic singularities for $t = 0$. Suppose that, for general $t \in B$, the fibre $\mathfrak{X}_t$ admits a Chow decomposition of the diagonal (or, equivalently, $\mathfrak{X}_t$ is universally* CH_0 *trivial). Then the same holds for any smooth projective birational model $\widetilde{\mathfrak{X}}_0$ of the special fibre $\mathfrak{X}_0$.*

This result can be rephrased in more convenient terms using the non existence formalism. Recall the following ubiquitary terminology: let $\mathfrak{X}$ be an algebraic variety:

- a property $\mathbb{P}$ holds for *general* points in $\mathfrak{X}$ if $\mathbb{P}$ holds outside a Zariski closed set $Z \subseteq \mathfrak{X}$;
- a property $\mathbb{P}$ holds for *very general* points in $\mathfrak{X}$ if $\mathbb{P}$ holds outside the union of a countable family of proper Zariski closed sets $Z_n \subseteq \mathfrak{X}$.

In particular, we can talk about properties of (very) general hypersurfaces of fixed degree d in $\mathbf{P}^n_k$, as the set of these varieties forms a projective space.

Corollary 3.2. *Let $f : \mathfrak{X} \longrightarrow B$ be a proper, flat projective morphism, where B is a smooth curve. If $\mathfrak{X}_0$ is at worst nodal and the desingularisation $\widetilde{\mathfrak{X}}_0$ of the special fibre $\mathfrak{X}_0$ has no Chow decomposition of the diagonal, then for a very general $t \in B$, the fibre $\mathfrak{X}_t$ has no Chow decomposition of the diagonal.*

The proof of these results can be found [21].

Remark 3.3. Theorem 3.1 does not hold without any restricion on the singularities of the special fibre at all. Indeed, the proof is split in two parts: the first parts uses an intersection theory argument to show that the decomposition of the diagonal in $\mathfrak{X}_t$ can be replicated in $\mathfrak{X}_0$ and holds without hypotheses on the fibres. The second part attempts to compare $\widetilde{\mathfrak{X}}_0$ with $\mathfrak{X}_0$ and it is here that one needs to consider only "mildly singular" cases.

Remark 3.4. The assumption that f is of relative dimension ≥ 2 is necessary: the disjoint union of two $\mathbf{P}^1$ does not admit a Chow decomposition on the diagonal.

In the applications, the specialisation theorem is used following this scheme:

- suppose X is a projective variety (with mild singularities, as stated in the hypotheses) and suppose that a smooth birational model $\widetilde{X}$ of X has no Chow decomposition of the diagonal (or it is not universally CH_0 trivial). This realistically can happen if $\widetilde{X}$ has some non-trivial invariant (like the Brauer group);
- realise X as special fibre of a proper, flat morphism $\mathfrak{X} \longrightarrow B$;
- by specialisation, the very general fibre $\mathfrak{X}_t$ has no Chow decomposition of the diagonal (or it is not universally CH_0 trivial);
- as seen in Theorem 2.4, if $\mathfrak{X}_t$ is smooth, this implies that $\mathfrak{X}_t$ is not stably rational.

In order to widen the range of application of this method, the hypotheses on the singularities of the special fibre have been weakened in [7], in a way we shall now explain. We first introduce a *relative* notion of universal CH_0-triviality.

Definition 3.5. A projective morphism $f : X \longrightarrow Y$ of varieties defined over a field k is *universally* CH_0*-trivial* if, for any field extension F/k, the push-forward map $f_* : \mathrm{CH}_0(X_F) \longrightarrow \mathrm{CH}_0(Y_F)$ is an isomorphism.

Therefore, we have the following "local" specialisation result.

Theorem 3.6. *Let A be a discrete valuation ring, $K = \mathrm{Quot}(A)$ its field of fractions and k its residue field, which we assume to be algebraically closed. Let $\mathfrak{X} \longrightarrow \mathrm{Spec}(A)$ be a proper, flat morphism with integral geometric fibres. Let us call X the generic fibre over K and Y the special fibre over k. Assume that Y admits a resolution of singularities $f : \widetilde{Y} \longrightarrow Y$ such that f is universally* CH_0*-trivial and assume one of the following:*

1. *X is smooth and $\widetilde{Y}$ has a zero-cyle of degree 1;*
2. *$X_{\bar{K}}$ is universally CH_0-trivial, for an algebraic closure $\bar{K}$ of K.*

Then $\widetilde{Y}$ is universally CH_0*-trivial.*

The usual application of this local theorem is as follows: suppose $f : \widetilde{Y} \longrightarrow Y$ is a resolution of singularities with f universally CH_0-trivial and suppose we know that, for any smooth birational model of Y some stable birational invariant (for instance, the Brauer group) is non-trivial. Then $\widetilde{Y}$ is not universally CH_0-trivial and, by the Theorem 3.6, every *smooth* variety X which specialises to Y is not universally CH_0-trivial. *A fortiori ratione*, this implies that X is not stably rational. The hypothesis of smoothness of X could also be replaced with the more general assumption that X has a resolution of singularities $\widetilde{X} \longrightarrow X$ with $\widetilde{X}$ smooth, projective and universally CH_0-trivial.

There follows, also, a global argument by using the point of view of decomposition of the diagonal.

Theorem 3.7. *Let B an integral k-scheme of finite type, where k is an infinite algebraically closed field of characteristic 0 and let $f : \mathfrak{X} \longrightarrow B$ be a proper, flat projective morphism. If there is a point $t_0 \in B(k)$ such that $\mathfrak{X}_{t_0}$ has no Chow decomposition of the diagonal, then for a very general point $t \in B(k)$, the fibre $\mathfrak{X}_t$ has no Chow decomposition of the diagonal.*

In particular, the two results are coupled to obtain the following useful Corollary.

Corollary 3.8. *Let B an integral k-scheme of finite type, where k is an infinite algebraically closed field of characteristic 0 and let $f : \mathfrak{X} \longrightarrow B$ be a flat projective morphism. If there is a point $t_0 \in B(k)$ such that $\mathfrak{X}_{t_0}$ is smooth and not universally* CH_0*-trivial, then for a very general point $t \in B(k)$, the fibre $\mathfrak{X}_t$ is not universally* CH_0*-trivial.*

4. Examples.

In this section we aim to present some notable applications of the above Theorem 3.1. We shall start presenting the Artin-Mumford example, then we will use this construction to show how it is employed to prove that the very general quartic 3-fold is not stably rational. For convenience, we work over $k = \mathbf{C}$, even if most of the constructions could be carried out over $\overline{\mathbf{Q}}$.

Recall the formal construction of double solids, following [5]. Let $S \subseteq \mathbf{P}^3$ be a hypersurface of *even* degree $2n$, having at worst ordinary nodes (singularities of multiplicity at most 2). Then S is a Cartier divisor and can be expressed as zero set of a global section σ of $\mathscr{O}_{\mathbf{P}^3}(2n)$. Definie

$$E = \operatorname{Tot}(\mathscr{O}_{\mathbf{P}^3}(n)) := \mathbf{Spec}\,\operatorname{Sym}\mathscr{O}_{\mathbf{P}^3}(-n)$$

namely the total space of the line bundle $\mathscr{O}_{\mathbf{P}^3}(n)$. Let $q : \mathscr{O}(n) \longrightarrow \mathscr{O}(2n)$ the square map (whose action on sections is $\tau \mapsto \tau^2$) and define

$$V := \{x \in E \mid q(x) \in \sigma(\mathbf{P}^3)\}$$

This defines a branched double cover $\pi : V \longrightarrow \mathbf{P}^3$: for each $p \in \mathbf{P}^3 \setminus S$ we have $\sigma(p) \neq 0$, hence $q^{-1}(p)$ has two values; however, π is branched along the whole S, as σ vanishes there. Furthermore, the possible singular points of V are exactly the inverse images of the possible singular points of S.

Definition 4.1. The 3-fold V is called *double solid branched along S.*

In the applications, it is often more useful to work with a coordinate representation of V. Indeed, if $S \subseteq \mathbf{P}^3$ is the surface cut out by the homogeneous equation $\sigma(X_0, X_1, X_2, X_3) = 0$, then V can be expressed as the locus of the points $[X_0 : X_1 : X_2 : X_3 : X_4] \in \mathbf{P}^4$ satisfying the equation

$$X_4^2 = \sigma(X_0, X_1, X_2, X_3)$$

Note that this is an affine equation, since the polynomial $X_4^2 - \sigma(X_0, X_1, X_2, X_3)$ is not homogeneous.

Otherwise, V can be seen as the birational model for the field $k(S)(\sqrt{s})$, namely the field of rational functions on S adjoint with a square root of its equation s. When s is a square in $k(S)$, the resulting double covering consists trivially of two identical, disjoing copies and it is said *to split.*

4.1. **The Artin-Mumford quartic double solid.**

Artin and Mumford in [1] constructed a quartic hypersurface in $\mathbf{P}^4 := \mathbf{P}^4_{\mathbf{C}}$ which is unirational but not stably rational. The example can be viewed both as double solid ramified over a particular cubic nodal curve. We sketch here the geometric construction.

Let $C \subseteq \mathbf{P}^2$ be a smooth conic defined by a homogeneous equation

$$f(X_0, X_1, X_2) = 0$$

and then let $E_1, E_2 \subseteq \mathbf{P}^2$ be two smooth cubic curves, defined by equations

$$g_1(X_0, X_1, X_2) = 0, \quad g_2(X_0, X_1, X_2) = 0$$

respectively. Suppose that g_1, g_2 are chosen such that each E_j is tangent to C at three distinct point *and* such that $E_1 \cap E_2$ consists of nine pairwise distinct points (of multiplicity one), also different from the previous ones. Recall the following classical result.

Lemma 4.2. *For any* 6 *points in linearly general position there passes an unique cubic curve.*

Proof. See ([12]). □

Therefore, there is a third cubic E meeting all the points in $E_1 \cap C$ and $E_2 \cap C$, defined, say, by a homogeneous equation

$$g(X_0, X_1, X_2) = 0$$

Hence the sextic homogeneous polynomial $g_1g_2+g^2$ vanishes at the six points of $(E_1 \cap C) \cup (E_2 \cap C)$ so it must vanish on the whole conic C. In other words, there is a quartic polynomial h such that

$$fh = g_1g_2 + g^2$$

This means, in particular, that $g_1g_2 = fh - g^2$, hence the sextic defined by the equation $fh - g^2 = 0$ is the union $E_1 \cup E_2$ of two cubics meeting transversally.

Now, define $B \subseteq \mathbf{P}^3$ as the quartic surface defined by the following homogeneous equation:

$$\begin{aligned}(4.1)\quad \varphi(X_0, X_1, X_2, X_3) := X_3^2 f(X_0, X_1, X_2)+ \\ - X_3 g(X_0, X_1, X_2) - h(X_0, X_1, X_2) = 0\end{aligned}$$

where f, g, h are the polynomials defined before. Now, clearly B has a node at $p_1 = [0 : 0 : 0 : 1]$: in fact, all the derivatives vanish at p_1, as f, g, h are homogeneous of positive degree, while

$$\frac{\partial^2 \varphi}{\partial X_3^2} = 2f$$

does not vanish at p_1. Moreover, note that the projection $\pi : B \backslash \{p_1\} \longrightarrow \mathbf{P}^2$ away from p_1 induces a morphism $\widetilde{\pi} : \widetilde{B} \longrightarrow \mathbf{P}^2$, where $\widetilde{B} = \mathrm{Blow}_{p_1}(B)$, which is a double cover ramified along the sextic g^2+fh, hence along $E_1 \cup E_2$. This imposes additional nine ordinary double points to B.

Now, consider $V \subseteq \mathbf{P}^4$ as the solid defined by the equation

$$X_4^2 = \varphi(X_0, X_1, X_2, X_3)$$

This is clearly a double solid branch along B as defined above, namely a double cover of $\mathbf{P}^3$ whose branch locus is the surfaces B. By the above remark on the singularities of B, it follows that V has 10 ordinary double points.

Artin and Mumford showed the following.

Theorem 4.3. *Any smooth birational model of the double solid V constructed above is not stably rational.*

The technique employed aimed to show that $H^3(X(\mathbf{C}), \mathbf{Z})$ has a non-trivial torsion element; in this case, however, this is equivalent to determine a non trivial 2-torsion element in $\mathrm{Br}(X)$. We will show this in a subsequent section, using techniques from the conic fibrations formalism.

Now, the above Theorem 4.3 shows stable irrationality of V, but indeed it is not hard to check that V is unirational.

Theorem 4.4. *Any smooth birational model of the double solid V constructed above is unirational.*

Proof. Let X be the desingularisation of V, obtained blowing up the nodes. Now, in suitable affine coordinates $X_0 = 1$, the equation of V is

$$X_4^2 = X_3^2(X_1^2 - X_2) - X_3 g(X_1, X_2) - h(X_1, X_2)$$

Now let us consider the affine double cover W defined, in $\mathbf{P}^5$, by

$$X_5^2 = X_1^2 - X_2$$

Hence, in W, the new 3-fold has equation

$$X_4^2 = X_3^2 X_5^5 - X_3 g(X_1, X_1^2 - X_5^2) - h(X_1, X_1^2 - X_5^2)$$

and this is a rational variety, via the map $\Phi : W \dashrightarrow \mathbf{A}^3 = \mathrm{Spec}(\mathbf{C}[y_1, y_2, y_3])$ defined as

$$\Phi(X_0 : \cdots : X_5) := (X_1, X_5, X_4 - X_5 X_3)$$

In fact, if we aim to calculate the fibre of Φ above $(y_1, y_2, y_3) \in \mathbf{A}^3$, let us set

$$X_1 = y_1, X_5 = y_2, X_4 = y_3 + y_2 X_3$$

and they need to satisfy the equation

$$y_3^2 + y_2^2 X_3^2 + 2y_2 y_3 X_3 = y_2^2 X_3^2 - g(y_1, y_1^2 - y_2^2) X_3 - h(y_1, y_1^2 - y_2^2)$$

which can be rewritten as

$$(2y_2 y_3 + g(y_1, y_1^2 - y_2^2)) X_3 = h(y_1, y_1^2 - y_2^2)$$

and has indeed a solution if

$$2y_2 y_3 + g(y_1, y_1^2 - y_2^2) \neq 0$$

This is sufficient to prove that W is birational to $\mathbf{P}^3$.

Hence, we have determined a *dominant* rational map $\mathbf{P}^3 \dashrightarrow V$, which extends to a dominant rational map $\mathbf{P}^3 \dashrightarrow X$, proving then the desired result. □

4.2. **The very general quartic hypersurface.** Quartic hypersurfaces have been successfully studied since the late XIX century but there are still many unsolved questions in low dimension. Indeed, in 1936 Ugo Morin proved the following result ([14]).

Theorem 4.5. *The generic quartic hypersurfaces $X \subseteq \mathbf{P}_k^n$ is unirational if $n \geq 7$, over any field k.*

This result was extended to $n = 6$, still by Morin in 1952 ([15]), and to $n = 5$, by Alberto Conte and Jacob Murre in 1998, using a result by Beniamino Segre from 1954 ([8]).

However, it is still unknown if the quartic 3-folds are all unirational or not. Some of them, actually, are unirational. For instance, the quartic $X \subseteq \mathbf{P}^3_{\mathbf{C}}$ defined by the homogeneous equation

$$X_0^4 + X_1^4 + X_2^4 + X_3^4 + X_0X_4^3 + X_3^3X_4 - 6X_1^2X_2^2 = 0$$

is smooth and unirational; this construction is due to Beniamino Segre ([17]). This same quartic hypersurfaces was proved to be irrational in 1971 by Iskovskikh and Manin, using the method of "birational rigidity": they proved that every birational automorphism of X extends to an automorphism of X; but the birational automorphism group of rational varieties is huge, while X has not many (regular) automorphisms.

Here we will concentrate on sketching the proof of the fact that the very general quartic 3-fold is not stably rational. This is done using the generalised version of the specialisation theorem, namely the Theorem 3.6. We will assume the ground field to be $\mathbf{C}$, but the whole construction could be presented entirely over $\overline{\mathbf{Q}}$.

First, let us consider the quartic hypersurface $Y \subseteq \mathbf{P}^4$ which is cut out by the equation

$$X_0^2X_4^2 = X_3^2 f(X_0, X_1, X_2) + X_3 g(X_0, X_1, X_2) + h(X_0, X_1, X_2)$$

where f, g, h are the same homogeneous polynomials as in (4.3). Thus, Y is birational to the Artin-Mumford double solid, by taking $X_0 = 1$. In particular, for any resolution of singularities $f : \widetilde{Y} \longrightarrow Y$, we have that $\mathrm{Br}(\widetilde{Y})[2] \neq 0$.

Now, the singularities of Y are more complicated than ordinary double points, so we can not apply Theorem 3.1 directly. However, it can be shown that there exists a resolution f which is universally CH_0-trivial ([7]).

Hence, any smooth variety that specialises to Y can not be stably rational. More precisely, suppose that $\mathscr{Q} \simeq \mathbf{P}^{\binom{4+4}{4}-1} = \mathbf{P}^{69}$ is the projective space of quartics hypersurfaces in $\mathbf{P}^4$. Let

$$f : \mathfrak{X} \longrightarrow \mathbf{P}^{69}$$

be the universal family of quartics: this is obtained as follows: let

$$\mathfrak{X} := V(\sigma) \subseteq \mathbf{P}^4 \times \mathbf{P}^{69}$$

be the zero-set of a global section σ of $\mathscr{O}_{\mathbf{P}^4 \times \mathbf{P}^{69}}((4, 1))$, with coordinate representation

$$\sigma := \sum_{i_0 + \ldots + i_4 = 4} a_{i_0,\ldots,i_4} X_0^{i_0} \cdots X_4^{i_4}$$

The map f is then the projection on the second factor: each fibre fixes a choice of coefficients $a_{i_0,\ldots,i_4}$ and thus defines a quartic hypersurface via the vanishing of σ in $\mathbf{P}^4$. Now, let $\mathscr{U} \subseteq \mathbf{P}^{69}$ be the open set of smooth quartics and let $\mathfrak{m}_Y \in \mathscr{W} := \mathbf{P}^{69} \setminus \mathscr{U}$ be the $\overline{\mathbf{Q}}$-point corresponing to the quartic Y.

Let $L \subseteq \mathbf{P}^{69}$ be a line containing $\mathfrak{m}_Y$ such that $L \nsubseteq \mathscr{W}$. Now let us apply the local specialisation theorem to the local ring $A := \mathscr{O}_{L,\mathfrak{m}_Y}$: the scheme $\mathrm{Spec}(A)$ parametrises quartics with coefficients chosen in L and the above family f defines a flat family $f : \mathfrak{X} \longrightarrow \mathrm{Spec}(A)$, whose special fibre is Y. By the theorem, the geometric generic fibre $\mathfrak{X}_{\overline{\mathbf{C}(t)}}$ is not stably rational; this corresponds to a smooth quartic in $\mathbf{P}^4$. But then, for any $p \in L(\mathbf{C})$ that is not defined over $\overline{\mathbf{Q}}$, the corresponding quartic $\mathfrak{X}_p$ is isomorphic to $\mathfrak{X}_{\overline{\mathbf{C}(t)}}$.

This enables us to produce many smooth quartics that are not stably rational and, thus, it will allow us to invoke the global specialisation theorem, in the form of Corollary 3.8:

> in the universal family of quartics $\mathfrak{X} \longrightarrow \mathbf{P}^{69}$ there are smooth fibres $\mathfrak{X}_{t_0}$ which are not stably rational. By Theorem 2.4, this means that $\mathfrak{X}_{t_0}$ is not universally CH_0-trivial and, by specialisation, this imply that the very general fibre $\mathfrak{X}_t$ is not universally CH_0-trivial. Therefore, the very general fibre is not stably rational.

5. Brauer group of varieties and fields.

In this section we shall introduce some theoretical notions about Brauer groups, which we have already mentioned being involved as obstruction to stable rationality. The proofs of the stated results can be found, together with many other details, in [11].

5.1. **Quaternion algebras.**

Let us start with a general definition. In what follows, unless otherwisely stated, we will work with a general field k of characteristic $\mathrm{char}\, k \neq 2$.

Definition 5.1. Let k be a field and let $a, b \in k^\times$. The *quaternion algebra* (a, b) is the 4-dimensional k-algebra generated by symbols x, y subject to relations $x^2 = a, y^2 = b$ and $xy = -yx$.

In a more concrete way, (a, b) is a k-vector space with basis given by $\{1, x, y, xy\}$ and with an inner product ruled by the above relations. By definition, it is clear that the isomorphism class of quaternion algebras (a, b) depends only on the classes of a, b in $k^\times/(k^\times)^2$. Indeed, if $a = u^2\alpha$ and $b = v^2\beta$, the substitution $x \mapsto ux$, $y \mapsto vy$ yelds an isomorphism $(\alpha, \beta) \simeq (u^2\alpha, v^2\beta) = (a, b)$. In particular, this shows that $(a, b) \simeq (b, a)$.

Every quaternion algebra is endowed with an involution$^-$: $(a, b) \longrightarrow (a, b)$, which can be defined in the following way. Let us call $q \in (a, b)$ a *pure quaternion* if $q^2 \in k$ but $q \notin k$. Each $q \in (a, b)$ can be written uniquely as $q = q_0 + q_1$ where $q_0 \in k$ and q_1 is a pure quaternion. Therefore, define

$$\bar{q} := q_0 - q_1$$

This also allows us to define a *norm* over (a, b), by setting $N : (a, b) \longrightarrow k$ as

$$N(q) := q\bar{q}$$

Note that

$$N(q) = (q_0 + q_1)(q_0 - q_1) = q_0^2 - q_1^2 \in k$$

so the norm is well defined.

Example 5.2. (1) The classical example of quaternion algebra is obviously the algebra of Hamilton quaternions, namely the $\mathbf{R}$-vector space $\mathbf{H} = \langle 1, i, j, ij \rangle$ with an inner product defined by the relations

$$i^2 = j^2 = -1, ij = -ji$$

Following our notations, it is clear that $\mathbf{H} = (-1, -1)$.

(2) A more refined example is given by the vector space $M(2, k)$ of square matrices. This is turned into a quaternion algebra $(-1, b)$, for each $b \in k^\times$, setting the following basis elements:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$$

and defining an isomorphisms $(1, b) \simeq M(2, k)$ via setting $1 \mapsto I, x \mapsto X, y \mapsto Y, xy \mapsto XY$.

Definition 5.3. A quaternion algebra (a, b) over k is called *split* if there exists an isomorphism of k-algebras $(a, b) \simeq M(2, k)$.

Every non-split quaternion algebra (a, b) is a *division algebra*, namely it is such that each non-zero element has a multiplicative two-sided inverse. Existence of division algebra depends strongly on the base field: if $k = \mathbf{R}$, then the only finite-dimensional division algebras over it are, up to isomorphism, $\mathbf{R}$ itself, its algebraic closure $\mathbf{C}$ and the quaternion algebra $\mathbf{H}$. If, instead, k is algebraically closed, then every division algebra over k is isomorphic to k itself.

We have the following characterisation of splitness over a field.

Proposition 5.4. *Let (a, b) be a quaternion algebra over k. The following properties are equivalent:*

1. *(a, b) is split;*
2. *(a, b) is not a division algebra;*
3. *the norm map of (a, b) has a non-trivial zero;*
4. *b is a norm in the field extension $k(\sqrt{a})/k$;*
5. *a os a norm in the field extension $k(\sqrt{b})/k$.*

To each quaternion algebra (a, b) over k, we can attach a projective plane quadric curve $C(a, b)$ defined over k, cut out by the following equation:

$$aX^2 + bY^2 - Z^2 = 0$$

where X, Y, Z are coordinates for $\mathbf{P}^2_k$. We have the following important result.

Proposition 5.5. *A quaternion algebra (a, b) is split if and only if the associated conic $C(a, b)$ has a k-rational point.*

Recall that a smooth projective curve over a field k is rational if and only if it is isomorphic to the projective line, and if and only if it has a k-rational point.

5.2. **Central simple algebras.** Recall that the *centre* $Z(A)$ of an algebra A is the set of the elements $x \in A$ commuting with every other element in A. If $Z(A) = k$, the algebra A is said to be *central.* It can be proved that every 4-dimensional central division algebra D over a field k is isomorphic to a quaternion algebra; the key point in proving this characterisation is that D contains a quadratic extension $k(\sqrt{a})/k$ amd $D \otimes_k k(\sqrt{a})$ is split over $k(\sqrt{a})$.

More precisely, if can be proved that a k-algebra A is isomorphic to a quaternion algebra (a, b) if and only if $A \otimes_k k(\sqrt{a})$ is split over $k(\sqrt{a})$. There is a whole class of algebras which can be described as those algebras splitting over a suitable extension of the base field.

Definition 5.6. A ring is *simple* if it has no non-trivial two-sided ideals.

The class we are interested into is the class of central algebras over a field k which are simple (as rings). For instance, every division algebra D is simple and, since $Z(D)$ is a field, D is central simple over $Z(D)$; every non-split quaternion algebra is of this kind. Moreover, the matrix ring $M(n, D)$ over any division algebra D is simple; the center of $M(n, D)$ is a copy of $Z(D)$ formed by scalar matrices (and it is a field), so $M(n.D)$ is central simple over $Z(D)$.

There exists an important characterisation theorem for simple algebras, which we state below.

Theorem 5.7. (Wedderburn) *Let A be a finite-dimensional simple algebra over a field k. Then there exist an integer $n \geq 1$ and a divison algebra $D \supset k$ such that $A \simeq M(n, D)$. Finally, D is unique up to isomorphism.*

If k is algebraically closed, then the only divison algebra containing k is k itself, so every finite-dimensional simple algebra over it is isomorphic to $M(n, k)$ for some n. This fact gives the idea for a different definition for central simple algebras.

Proposition 5.8. *Let k be a field and A a finite-dimensional k-algebra. Then A is central simple if and only if there exist an integer $n \geq 1$ and a finite field extension K/k such that $A \otimes_k K$ is isomorphic to $M(n, K)$.*

The field extension K/k of the previous theorem is called *splitting field* of the central simple algebra A; the commonly used terminology is that A splits over K to mean that $A \otimes_k K \simeq M(n, K)$. Moreover, the k-dimension of a central simple algebra A is a square, and $\sqrt{\dim_k A}$ is called the *degree* of A. The following results guarantees some nice conditions of the splitting field.

Theorem 5.9. (Noether, Köthe) *The splitting field of a central simple algebra is a separable extension.*

Since every finite separable field extension is contained into a finite Galois extension, the above results imply that a finite-dimensional k-algebra A is central simple if and only if it splits over a finite Galois extension K/k.

5.3. **Algebraic Brauer group.** We finally define the main object of interest in this chapter using the ideas recalled in the previous sections.

Lemma 5.10. *If A, B are two central simple k-algebras split over K, then so is $A \otimes_k B$.*

The above Lemma leads to the following definition.

Definition 5.11. Let A, B be two central simple k-algebras. Then A and B are said to be *Brauer equivalent* if there exist integers $m, n > 0$ such that $A \otimes_k M(m,k) \simeq B \otimes_k M(n,k)$.

We can make the definition above work as an equivalence relation. Call $\mathscr{C}(K/k,n)$ the set of central simple k-algebras of degree n which are split over a finite Galois extension K/k. Then Brauer equivalence defines an equivalence relation on the union of sets $\mathscr{C}(K/k,n)$ running on positive integers: if A, B and B, C are pairwise Brauer equivalent, we have isomorphisms

$$A \otimes_k M(n,k) \simeq B \otimes_k M(m,k), \quad B \otimes_k M(p,k) \simeq C \otimes_k M(q,k)$$

and

$$A \otimes_k M(np,k) \simeq B \otimes_k M(p,k) \simeq C \otimes_k (q,k)$$

Every equivalence class is called *Brauer class*.

Remark 5.12. By definition, each Brauer class contains an unique division algebra up to isomorphism. Then it is natural to say that the Brauer equivalence classifies division algebras over the ground field. Moreover, if A and B are two Brauer-equivalent algebras with same dimension, then by Wedderburn theorem they are even isomorphic.

We will denote $\mathrm{Br}(K/k)$ the set of all the Brauer classes and $\mathrm{Br}(k)$ the union of all $\mathrm{Br}(K/k)$ amongst all the finite Galois extensions of k. Since tensor product manifestly preserves Brauer equivalence of k-algebras, the sets above defined have a natural inner operation.

Proposition 5.13. *The sets $\mathrm{Br}(K/k)$ and $\mathrm{Br}(k)$ have an abelian group structure with the operation induced by tensor product of k-algebras.*

The set $\mathrm{Br}(K/k)$ equipped with the tensor product is called *Brauer group of K relative to k*, while $\mathrm{Br}(k)$ is the *(absolute) Brauer group of k*. The triviality of division algebras over algebraically closed fields leads to the following easy but important result.

Lemma 5.14. *Let k be an algebraically closed field. Then $\mathrm{Br}(k) = 0$.*

Proof. Since $\mathrm{Br}(k)$ parametrises central simple division algebras over k, it is enough to prove that no such algebra is non-trivial. Indeed if D is a division algebra over k then for each $d \in D$ the field $k[d]$ is a finite extension of k, which must be trivial by algebraic closure. □

Example 5.15. For instance, $\mathrm{Br}(\mathbf{R}) = \{\mathbf{R}, \mathbf{H}\} \simeq \mathbf{Z}/2$ by what we have seen previously. Note that $\mathbf{C}$ is a simple division algebra over $\mathbf{R}$, but it is not central, so it does not belong to $\mathrm{Br}(\mathbf{R})$.

5.4. **Brauer group and Galois cohomology.** There is an useful description of the Brauer group which uses some Galois cohomology groups.

Let K/k be a Galois field extension and recall that the Galois group $\mathrm{Gal}(K/k)$ is a profinite group:

$$\mathrm{Gal}(K/k) = \varprojlim (\mathrm{Gal}(L/k))_{L \subseteq K}$$

where L/k varies amongst all finite Galois sub-extensions of K/k. In particular, if k_{sep} is a separable closure of k, we can define the absolute Galois group $\mathrm{Gal}(k) := \mathrm{Gal}(k_{\mathrm{sep}}/k)$ and, for every continous $\mathrm{Gal}(k)$-module M, we can define the *Galois cohomology groups* as

$$H^p(k, M) := H^p(\mathrm{Gal}(k), M) := \varinjlim H^p(\mathrm{Gal}(K/k), M^{U_{K/k}}))$$

where K/k is a finite Galois sub-extension of k_{sep}/k and $U_{K/k}$ is a standard open set in $\mathrm{Gal}(k)$ with respect to the profinite topology, namely such that $\mathrm{Gal}(K/k) \simeq \mathrm{Gal}(k)/U_{K/k}$.

We recall here the following important result.

Proposition 5.16. (Kummer theory) *Let $m > 0$ be an integer prime to* $\operatorname{char} k$. *Denote with μ_m the group of m-th roots of unity in a separable closure k_{sep} of k, with the structure of continuous* $\mathrm{Gal}(k)$*-module given by the usual action of* $\mathrm{Gal}(k)$ *on k_{sep}. Therefore*

$$H^1(k, \mu_m) \simeq k^\times/(k^\times)^m$$

The above general setting can be fruitfully used to describe the Brauer group.

Theorem 5.17. *Let K/k be a finite Galois extension. Therefore*

$$\mathrm{Br}(K/k) \simeq H^2(\mathrm{Gal}(K/k), K^\times)$$

If moreover k_{sep} is a separable closure of k,

$$\mathrm{Br}(k) \simeq H^2(k, (k_{\mathrm{sep}}^\times))$$

One important consequence of this result is that the Brauer group is a torsion group, as Galois cohomology groups are so. We will be particularly interested in the m-torsion part of the Brauer group, which also can be described by means of Galois cohomology.

Corollary 5.18. *Let m be a positive integer, prime to* $\operatorname{char} k$. *Then we have the isomorphism*

$$\mathrm{Br}(k)[m] \simeq H^2(k, \mu_m)$$

where μ_m is the (multiplicative) group of m-th roots of unity in k_{sep}, equipped with the standard $\mathrm{Gal}(K/k)$*-action.*

Remark 5.19. For convenience, we omitted to explain what happens if we consider the m-th torsion part with m not prime to $\operatorname{char} k$. There are theorems replacing the above results (Artin-Schreier theory) and some of the notable characterisations can still be achieved.

5.5. **Cohomological Brauer group.** The notion of Brauer group can be generalised to arbitrary schemes, with a construction due to Grothendieck which involves étale cohomology. We don't need many details here, so the construction is only sketched; the standard reference is [13]

Let X be a k-scheme. The group scheme $\mathbf{G}_m$ (which can be thought as the group scheme associated to the multiplicative group μ_m of m-th roots of unit in a separable closure of k) defines a sheaf on X for the étale topology. Thus, the following definition is meaningful.

Definition 5.20. The *cohomological Brauer group* of X is defined as

$$\operatorname{Br}(X) := H^2_{\text{ét}}(X, \mathbf{G}_m)$$

The torsion part of this group can be reconstructed, in many cases, using Morita equivalences of Azumaya algebras over X, in a similar fashion as we did for central simple algebras. When X is a regular k-variety, $\operatorname{Br}(X)$ is a torsion group and it is a subgroup of $\operatorname{Br}(k(X))$.

Now, let C a smooth conic over a field k. Suppose that $C(k) = \varnothing$, so that C defines a non-split quaternion algebra over k and, hence, a non-trivial Brauer class $\alpha_C \in \operatorname{Br}(k)$. By properties of étale cohomology, we have that

$$\operatorname{Br}(k) = H^2(k, \mathbf{G}_m) = H^2_{\text{ét}}(\operatorname{Spec}(k), \mathbf{G}_m)$$

so the natural morphism $C \longrightarrow \operatorname{Spec}(k)$ induces a pullback map

$$\operatorname{Br}(k) \longrightarrow H^2_{\text{ét}}(C, \mathbf{G}_m) = \operatorname{Br}(C)$$

Theorem 5.21. *In the above setting, if* $\operatorname{char} k \neq 2$ *and* -1 *is a square in* k*, there is an exact sequence*

$$0 \longrightarrow \mathbf{Z}/2 \longrightarrow \operatorname{Br}(k)[2] \longrightarrow \operatorname{Br}(C)[2] \longrightarrow \mathbf{Z}/2 \longrightarrow 0$$

Moreover, any class not in the image of $\operatorname{Br}(k)[2] \longrightarrow \operatorname{Br}(C)[2]$ *is* 4*-torsion, that is it belongs to the image of* $\operatorname{Br}(k)[4] \longrightarrow \operatorname{Br}(C)[4]$.

Concretely, our interest in the Brauer group is motivated by the fact that it can be used to obstruct stable rationality by means of the point of view of universal CH_0-triviality or Chow decomposition of the diagonal. We have, indeed, the following result (see [16]).

Theorem 5.22. *Let X be a smooth projective variety over a field k and assume that X has a Chow decomposition of the diagonal. Then, for any field extension L/k, the natural map* $\operatorname{Br}(L) \longrightarrow \operatorname{Br}(X_L)$ *is an isomorphism.*

If k is algebraically closed, the above result joint with Lemma 5.14 tells that

$$\operatorname{Br}(X) \simeq \operatorname{Br}(k) = 0$$

and recalling Theorem 2.4, we have the following statement.

Corollary 5.23. *Let X be a smooth, projective variety defined over an algebraically closed field k and suppose X is stably rational. Then* $\mathrm{Br}(X) = 0$.

In the special case $k = \mathbf{C}$, it can be proved that $\mathrm{Br}(X) \simeq \mathrm{tors}\, H^3(X(\mathbf{C}), \mathbf{Z})$, where the cohomology is the topological Betti cohomology. Hence, one recovers the original argument by Artin and Mumford, disproving stable rationality by showing non-triviality of the latter subgroup.

6. Residues and unramified cohomology

In this section we present some tools which may be useful to calculate $\mathrm{Br}(X)$ or, more precisely, to show its non-triviality.

Let K be the function field of an integral variety X defined over a field k such that $\mathrm{char}\, k \neq 2$. Upon this choice, -1 is a square in a chosen separable closure K_{sep} of K and then there is an isomorphism

$$\mu_2 \simeq \mathbf{Z}/2$$

Recall that Kummer theory 5.16 induces an isomorphism

$$H^1(K, \mathbf{Z}/2) \simeq K^\times/(K^\times)^2$$

Similarly, by Corollary 5.18,

$$H^2(K, \mathbf{Z}/2) \simeq \mathrm{Br}(K)[2] \tag{6.1}$$

Suppose D is a prime divisor in X, such that X is regular at the generic point of D. Note that D corresponds to an unique divisorial valuation ν_D over the field $K = k(X)$, with residue field $k(D)$, the function field of D and discrete valuation ring exactly $\mathscr{O}_{X,\eta_D}$ the local ring of X at the generic point of D. We want to define group morphisms

$$\partial^1_D : H^1(K, \mathbf{Z}/2) \longrightarrow H^0(k(D), \mathbf{Z}/2) = \mathbf{Z}/2$$
$$\partial^2_D : H^2(K, \mathbf{Z}/2) \longrightarrow H^1(k(D), \mathbf{Z}/2)$$

called *residue maps at D*. Let us procced in the following way: let $\bar{a} \in H^1(K, \mathbf{Z}/2) = K^\times/(K^\times)^2$ and define

$$\partial^1_D(\bar{a}) := \nu_D(a) \mod 2$$

Then, let $\alpha \in H^2(K, \mathbf{Z}/2)$; according to isomorphism (6.1), α can be represented by a symbol (a, b) for some $a, b \in K^\times$, corresponding to a plane smooth conic C in $\mathbf{P}^2_K$. We define

$$\partial^2_D(a, b) := (-1)^{\nu_D(a)\nu_D(b)} \left(a^{\nu_D(b)} b^{-\nu_D(a)}\right)_D \tag{6.2}$$

where $(-)_D$ indicates the square class in $k(D)^\times/(k(D)^\times)^2$. Recall indeed that $k(D)$ is a field extension of K. Note that if π is an uniformiser parameter in the valuation ring of ν_D, then ∂^2_D is determined uniquely by the value $\partial^2_D(\pi, u) = (u)_D$, where u is a unit in the valuation ring.

If X is not regular at the generic point of D, we also have an alternative description of ∂^1_D. In this case, the local ring $\mathscr{O}_{X,\eta_D}$ is not necessarily a discrete valuation ring. Suppose $\widehat{X} \longrightarrow X$ is the normalisation and suppose $D_1, \ldots, D_s$ are the irreducible components lying over D. Then each D_i defines a discrete divisorial valuation of $k(\widehat{X}) = K$ with residue field $k(D_i)$ and we define, for each $a \in L^\times$

$$\partial^1_D(\bar{a}) := \sum_{i=1}^{s} |k(D_i)/k(D)| \nu_{D_i}(a) \mod 2$$

where $|k(D_i)/k(D)|$ is the degree of the extension induced by the natural dominant morphism $D_i \to D$.

The above formulas can be rephrased in a more general setting. Let $\mathscr{P}(K/k)$ be the set of *places* on K, namely divisorial valuations over K which fix k; for each $\nu \in \mathscr{P}(K/k)$ we will denote with $k(\nu)$ the residue field of ν. Clearly formula (6.2) makes sense for any divisorial valuations $\nu \in \mathscr{P}(K/k)$, even if it has not necessarily centre in a divisor D of X.

Definition 6.1. Let X be an integral variety defined over a field k with $\operatorname{char} k \neq 2$ and let K be its function field. The *second unramified cohomology group* of K over k with coefficients in $\mathbf{Z}/2$ is defined as

$$H^2_{\mathrm{nr}}(K/k, \mathbf{Z}/2) := \bigcap_{\nu \in \mathscr{P}(K/k)} \ker(\partial^2_\nu)$$

Note that the definition of unramified cohomology depends on the ground field k: indeed, the divisorial valuations in $\mathscr{P}(K/k)$ are meant to be fixing k. In general, an element a of $H^2(K, \mathbf{Z}/2)$ is called *unramified* with respect to a certain class of places $\Lambda \subseteq \mathscr{P}(K/k)$ if $\partial^2_\nu(a) = 0$ for each $\nu \in \Lambda$.

It is immediate to see that unramified cohomology groups are birational invariants. We also have the following theorem.

Theorem 6.2. *Let X be a smooth projective variety over a field k and let $m > 0$ be an integer prime to* $\operatorname{char} k$*. Therefore*

$$H^2_{\mathrm{nr}}(k(X)/k, \mathbf{Z}/m) \simeq \operatorname{Br}(X)[m]$$

In particular, in the case $m = 2$ and $\operatorname{char} k \neq 2$, we find $\operatorname{Br}(X)[2]$. Recalling Corollary 5.23, this means that non-triviality of unramified cohomology obstructs the stable rationality of X.

Of course, in the definition of $H^2_{\mathrm{nr}}(K/k, \mathbf{Z}/2)$, it is impossible to check *every* place in $\mathscr{P}(K/k)$, so in practice one needs some result that restricts this set to an appropriate set of valuations corresponding to prime divisors over a *fixed* model of K. Such kind of results are implied by the so-called "cohomological purity", of which we will need the following geometric version.

Proposition 6.3. *Let k be a field such that* $\operatorname{char} k \neq 2$ *and let X be a smooth variety over k. Call $K = k(X)$ the field of rational functions of X. Then every element in $H^2(K, \mathbf{Z}/2)$ which is unramified with respect to divisorial valuations corresponding to prime divisors of X is also unramified* *with respect to* $\mathscr{P}(K/k)$*, namely all the divisorial valuations having centre on* X*.*

More details about this can be found in [6]. Finally, we shall recall the following result which shows the behaviour of Brauer classes with respect to their residues.

Proposition 6.4. *Let* k *be an algebraically closed field and let* B *be a smooth algebraic surface over* k*. Suppose that* $\operatorname{char} k(B) \neq 2$*; therefore if* $H^1_{\text{ét}}(B, \mathbf{Z}/2) = 0$*, we have the following exact sequence*

$$0 \longrightarrow \mathrm{Br}(B)[2] \longrightarrow \mathrm{Br}(k(B))[2] \xrightarrow{\oplus \partial^2} \bigoplus_{D \in \mathrm{Div}(B)} H^1(k(D), \mathbf{Z}/2) \xrightarrow{\oplus \partial^1}$$
$$\longrightarrow \bigoplus_{p \in B} \mathbf{Z}/2 \longrightarrow \mathbf{Z}/2 \longrightarrow 0$$

where the first arrow is induced by the restriction to the generic point of B *and the last arrow is the sum.*

7. Conic fibrations.

The machinery introduced in the previous sections is fruitfully applied to a particular class of varieties, *conic fibrations* or, more informally, *conic bundles*. Roughly speaking, a conic bundle is a proper, projective morphism whose fibres are conics. A more general definition leads to *quadric fibrations*, namely proper projective morphisms whose fibres are quadrics of arbitrary dimension. The advantages of studying conic bundles are roughly two:

(1) they are essentially easy (or at least easier than other varieties) to organise in families (namely, they"deform well" in flat families);
(2) there are good results about Brauer group and unramified cohomology of them.

We will show later what is the role of conic bundles in the stable Lüroth problem. Here we shall only introduce the basic definition and the fundamental examples.

Definition 7.1. Let X, Y be projective varieties defined over a field k such that $\operatorname{char} k \neq 2$. A *conic fibration* is a projective morphism $f : Y \longrightarrow X$ whose fibres are isomorphic to plane conics. More precisely,

(1) f factors through $Y \hookrightarrow \mathbf{P}^2_X$ and $\mathbf{P}^2_X \twoheadrightarrow X$;
(2) for each $p \in X$, the fibre Y_p is isomorphic to a conic in $\mathbf{P}^2_X$.

A *rational conic fibration* is a morphism $f : Y \longrightarrow X$ such that f is a conic fibration in a whole open set of X.

The fibres of a conic fibration f can have three different behaviours: they may be a smooth conic, a pair of lines and a double line.

Lemma 7.2. *If* X *and* Y *are integral, projective varieties, then* f *is a rational conic fibration if and only if the generic fibre is a smooth conic in* $\mathbf{P}^2_K$*, where* $K = k(X)$ *is the field of functions of the base* X*.*

Proof. Suppose f is a rational conic fibration; then there exists an open affine set $U \subseteq X$ such that $f : Y \longrightarrow U$ is a conic fibration. Hence $Y_\eta = Y \times_U \operatorname{Spec} K$ is a smooth conic on $\mathbf{P}^2_U$. But $\mathbf{P}^2_U = \mathbf{P}^2_k \times_{\operatorname{Spec}(k)} U$ and $U = \operatorname{Spec} K[X_1, \ldots, X_m]$ so $\mathbf{P}^2_U = \mathbf{P}^2_K$. Conversely, assume Y_η is a smooth conic on $\mathbf{P}^2_K$. By definition of generic point, there exists an open set $U \subseteq X$ such that Y_p is a smooth conic for each $p \in U$. □

In the applications, we almost always shall restrict to the notable case in which X is a rational, smooth variety. Indeed, to emphasize the fact that we are treating a special case, we will refer to these conic fibrations as *conic bundles*. Accordingly, a variety Y will have a conic bundle structure over $\mathbf{P}^m$ if there exists a projective morphism $f : Y \longrightarrow \mathbf{P}^m$ whose fibres are plane conics.

Conic bundles can be defined in a more explicit way, by means of zeros of suitable quadratic forms. Indeed many conic bundles can be constructed in this way: suppose that ψ is a form on $\mathbf{P}^2 \times \mathbf{P}^m$, of degree 2 in the first component, defined as

$$([X_0 : X_1 : X_2], [t_0 : \cdots : t_m]) \mapsto \psi_{[t_0:\cdots:t_m]}(X_0, X_1, X_2)$$

Then defining

$$Y := \{([X_0 : X_1 : X_2], p) \in \mathbf{P}^2 \times \mathbf{P}^m \mid \psi_p(X_0, X_1, X_2) = 0\}$$

we have a natural projection map

$$\begin{aligned} \pi : Y &\longrightarrow \mathbf{P}^m \\ ([X_0 : X_1 : X_2], p) &\mapsto p \end{aligned}$$

and it is clear that the fibres above each point $p \in \mathbf{P}^m$ are plane conics defined by the quadratic equation $\psi_p(X_0, X_1, X_2) = 0$. This elementary setting can be reinterpreted using vector bundles and scheme formalism.

Proposition 7.3. *Let $f : X \longrightarrow \mathbf{P}^m$ a conic bundle. Then*

1. *f is a flat morphism;*
2. *there exist a vector bundle $\mathscr{E}$, an integer n and a global section $\sigma \in H^0(\mathbf{P}^m, \operatorname{Sym}^2 \mathscr{E}(n))$ such that Y can be identified as the set of the zeros σ in $\mathbf{P}(\mathscr{E}) := \mathbf{Proj}\operatorname{Sym}(\mathscr{E})$;*
3. *there exists a sub-variety $\Delta \subseteq \mathbf{P}^m$, with at most nodal singularities, such that:*
   (a) *for each $p \in \mathbf{P}^m \setminus \Delta$, the fibre Y_p is smooth;*
   (b) *if $p \in \Delta$ is non-singular, the fibre Y_p has exactly a singular point;*
   (c) *if $p \in \Delta$ is singular, the fibre Y_p has a whole line of singular points.*

Proof. See ([3]) □

The locus Δ in the above result is called *discriminant divisor* of the conic bundle and has a remarkable role in our further investigations. As said, Δ

consists of those points p such that the conic Y_p is singular and this can happen in two different ways: Y_p may be a singular, reduced conic (*id est*, two crossing lines) or may be a singular but non-reduced conic (*id est*, a whole line of double points).

Remark 7.4. Of course, when a conic bundle is described by means of a quadratic form ψ, then Δ is the zero locus of $\det \psi$.

7.1. **The cubic 3-fold.** A classical example of conic bundle structure is given by a cubic 3-fold in $\mathbf{P}^4 := \mathbf{P}^4_{\mathbf{C}}$. We will use the following well-known result.

Lemma 7.5. *Let X be a smooth cubic 3-fold in $\mathbf{P}^4$. Therefore, X contains a 2-dimensional linear system of lines.*

The conic bundle structure on X is then given as following. Suppose L_0 is a line in X and consider the family of hyperplanes containing L_0: it is a 2-dimensional family up to scalars, so it can be thought as a $\mathbf{P}^2$. Then, let us consider the projection map $\pi : X \setminus L_0 \longrightarrow \mathbf{P}^2$ away from L_0. Since every hyperplane containing L_0 cuts out a residual conic, we have proved that π has fibres isomorphic to smooth conics. Now, calling $\widetilde{X}$ the blow-up of X along L_0, the map π extends to a morphism $\widetilde{\pi} : \widetilde{X} \longrightarrow \mathbf{P}^2$ whose fibres are isomorphic to smooth conics or a pair of distinct lines.

7.2. **The Artin-Mumford double solid as a conic bundle.** The example provided by Artin and Mumford, which we described in paragraph 4.1, can be viewed as a particular conic bundle over $\mathbf{P}^2$. Recall, firstly, that V is the double cover of $\mathbf{P}^3$ ramified along a nodal cubic surface B, which has exactly 10 ordinary double points.

Suppose $\mathbf{P}^2$ is a plane containing a node $\widetilde{p} \in V$ (corresponding to a node $p \in B$); the projection map away from that plane yelds a rational map $\pi : V \dashrightarrow \mathbf{P}^2$. We can think $\mathbf{P}^2$ as the linear system of lines through p in $\mathbf{P}^3$, and from this point of view π is a rational conic bundle.

Indeed, the lines in $\mathbf{P}^3$ through p meet B in three distinct points: p itself (with multiplicity 2) and two other points x, y (with multiplicity 1). Hence, the inverse image of such lines in V is a curve C, with a nodal singularity in $\pi^{-1}(p)$. Let $\widehat{C}$ be the normalisation of C: then $\widehat{C}$ is a double cover of $\mathbf{P}^1$, ramified at two points only (the inverse images of x, y as before). By the Riemann-Hurwitz formula,

$$2g(\widehat{C}) - 2 = 2(2g(\mathbf{P}^1) - 2) + (e_x - 1) + (e_y - 1)$$

so that

$$2g(\widehat{C}) = 2 - 4 + 2 = 0$$

hence $\widehat{C}$ is a rational quadric curve, hence a conic. This defines the rational conic bundle structure. Now, resolving the singularities of V, namely blowing up the nodes, we obtain a morphism $\widetilde{\pi} : \widetilde{V} \longrightarrow \mathbf{P}^2$ which has the desired conic bundle structure.

Let us determine the discriminant locus. Note that, given a point $x := (x_0 : x_1 : x_2) \in \mathbf{P}^2$, the fibre over x is given by the affine equation

$$X_4^2 = f(x_0, x_1, x_2)X_3^2 - g(x_0, x_1, x_2)X_3 - h(x_0, x_1, x_2)$$

representing a conic. This conic is singular if and only if $0 = (g^2 - fh)(x) = g_1 g_2(x)$, so if and only if x lies in the union of the two cubics $E_1 \cup E_2$, whose equation is $g_i = 0$ respectively. Moreover, the above conic is a double line $X_4^2 = 0$ if and only if f, g and h vanish simultaneously at x, and this would imply that x is a double point of $E_1 \cup E_2$ lying in the conic C defined by $f = 0$. There is no such point, so the discriminant curve Δ of π is the union of two cubic curves E_1, E_2 meeting transversely at 9 different points.

8. Conic bundles and unramified cohomology.

Now, we shall explain why conic fibrations are interesting when analysing the stable Lüroth problem. We will assume, at a first stage, that all the varieties are defined over a field k which is algebraically closed and such that $\operatorname{char} k \neq 2$.

Let $\pi : Y \longrightarrow X$ be a conic bundle, with Y smooth, projective variety defined over k. Note that the generic fibre Y_η is a smooth conic over $K := k(X)$, so it corresponds to a quaternion algebra over K and, hence, to Brauer class in $\mathrm{Br}(K)$ of order 2. So Y_η lies in $\mathrm{Br}(K)[2]$. Moreover, by Lemma 5.21, there is an exact sequence

$$0 \longrightarrow \mathbf{Z}/2 \longrightarrow \mathrm{Br}(K)[2] \xrightarrow{\tau} \mathrm{Br}(Y_\eta)[2] \longrightarrow \mathbf{Z}/2 \longrightarrow 0$$

By the theory of maximal orders (see [1]), it can be proved that the kernel of τ is generated by the Brauer class of Y_η.

Recall, also, that $\mathrm{Br}(Y)[2] = H^2_{\mathrm{nr}}(k(Y)/\mathbf{C}, \mathbf{Z}/2) = H^2_{\mathrm{nr}}(k(Y_\eta)/\mathbf{C}, \mathbf{Z}/2)$ by Theorem 6.2.

Example 8.1. (The Artin-Mumford conic bundle) Let $\pi : Y := \widetilde{V} \longrightarrow \mathbf{P}^2 := \mathbf{P}^2_{\mathbf{C}}$ the Artin-Mumford conic bundle with generic fibre $Q \subseteq \mathbf{P}_K$, where $K := k(\mathbf{P}^2)$. We shall show that $\mathrm{Br}(Y)[2] \neq 0$ or equivalently $H^2_{\mathrm{nr}}(k(Q)/\mathbf{C}, \mathbf{Z}/2) \neq 0$.

By Lemma 5.21, each class in $\mathrm{Br}(Q)[2]$ which is not 4-torsion comes from a class in $\mathrm{Br}(K)[2]$. Since

$$\mathrm{Br}(Q)[2] = H^2_{\mathrm{nr}}(k(Y)/K, \mathbf{Z}/2) \supseteq H^2_{\mathrm{nr}}(k(Y)/\mathbf{C}, \mathbf{Z}/2) = \mathrm{Br}(Y)[2]$$

we claim that any element $\xi \in \mathrm{Br}(Y)[2]$ come from a class $\xi_K \in \mathrm{Br}(K)[2]$.

Now, recall the exact sequence 6.4: applying it to the base $\mathbf{P}^2$ and using Lemma 5.21, we find the following commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
 & & 0 \arrow[d] & \mathrm{Br}(Y)[2] \arrow[d] \\
0 \arrow[r] & \mathbf{Z}/2 \arrow[r] & \mathrm{Br}(K)[2] \arrow[r, "\tau"] \arrow[d, "\oplus\partial^2"] & \mathrm{Br}(Q)[2] \arrow[d, "\oplus\partial^2"] \\
0 \arrow[r] & K \arrow[r] & \bigoplus_{D\in\mathrm{Div}(\mathbf{P}^2)} H^1(k(D),\mathbf{Z}/2) \arrow[r, "\pi^*"] \arrow[d, "\oplus\partial^1"] & \bigoplus_{C\in\mathrm{Div}(Y)_{\mathrm{d}}} H^1(k(C),\mathbf{Z}/2) \\
 & & \bigoplus_{p\in\mathbf{P}^2} \mathbf{Z}/2 &
\end{tikzcd}$$

Note that $\mathrm{Div}(Y)_{\mathrm{d}}$ is the set of prime divisors in Y which dominate the base $\mathbf{P}^2$ onto a divisor and π_D^* is the pullback map induced by restriction of π to such divisors. Note also that the rows and the columns are exact, either by definition or by the various Lemmas we have proved. Call α_Q the Brauer class in $\mathrm{Br}(K)[2]$ corresponding to the generic fibre Q; then $\langle\alpha_Q\rangle = \ker(\tau) \simeq \mathbf{Z}/2$. Finally, recall that, by Kummer theory, we have

$$H^1(k(D),\mathbf{Z}/2) \simeq k(D)^\times/(k(D)^\times)^2$$

Now, we know that the discriminant curve $\Delta \subseteq \mathbf{P}^2$ of π is the union of two cubics $E_1 \cup E_2$ without singular points. Note that each component E_i induces a non split (namely, not reducible) double covering via π, so indeed $\alpha_i = \partial^2_{E_i}(\alpha) \neq 0$, namely α is not a square in $k(E_i)$. By diagram chasing, it can be proved that

$$K = \ker\pi^* = \langle\alpha_1\rangle \oplus \langle\alpha_2\rangle \simeq (\mathbf{Z}/2)^2$$

Note that elements in $\mathrm{Br}(Y)[2] = H^2_{\mathrm{nr}}(k(Y)/\mathbf{C},\mathbf{Z}/2)$ are those elements in $\mathrm{Br}(Q)[2] = H^2_{\mathrm{nr}}(k(Y)/K,\mathbf{Z}/2)$ which are mapped to zero via the map $\oplus\partial^2_C$.

We now prove that elements of $\mathrm{Br}(Y)[2]$ come from $H^2(K,\mathbf{Z}/2)$ even though τ is not surjective. Suppose indeed $\xi \in \mathrm{Br}(Q)[2]$ does not lift to $H^2(K,\mathbf{Z}/2)$; then by Lemma 5.21, it lifts to an element $\xi' \in H^2(K,\mathbf{Z}/4) = \mathrm{Br}(K)[4]$. Since the map $\oplus\partial^2_D$ in the left column is injective, there exists one $D \in \mathrm{Div}(\mathbf{P}^2)$ such that $\partial^2_D(\xi') \in H^1(k(D),\mathbf{Z}/4)$ has order 4. But by the above description of the kernel of π^*, we must have $\pi^*(\partial^2_D(\xi')) \neq 0$. By commutativity, this means also that $\partial^2_C(\xi) \neq 0$ for $\pi(C) = D$ and, hence $\xi \notin \mathrm{Br}(Y)[2]$.

With a similar diagram chasing, one proves then that elements of $\mathrm{Br}(Y)[2]$ come from a subgroup H of K. This correspondence is an isomorphism up to a quotient of H, which is proved to be non-trivial. This finally proves that $\mathrm{Br}(Y)[2] \neq 0$ and, therefore, that the desingularisation of the Artin-Mumford double solid is not stably rational.

The strategy presented in the above Example can be generalised to conic bundles $\pi : Y \longrightarrow X$, where X is a smooth, projective rational surface. Indeed, there is a formula for the torsion part of $\mathrm{Br}(Y)$.

Theorem 8.2. (Colliot-Thélène) *Let X be a smooth, projective rational surface over $\mathbf{C}$ and let $K := k(X)$; let moreover Y be a smooth 3-fold equipped with a conic bundle structure $\pi : Y \longrightarrow X$. Suppose $\alpha \in \mathrm{Br}(K)[2]$ is the Brauer class corresponding to the generic fibre Y_η. Assume $\alpha \neq 0$ and assume the discriminant locus $\Delta \subseteq X$ has at worst quadratic singularities. Let $\Delta = \Delta_1 \cup \ldots \cup \Delta_n$ be the decomposition into irreducibile components and let $\alpha_{\Delta_i} := \partial^2_{\Delta_i}(\alpha)$ for each $i = 1, \ldots, n$. Consider the subgroup*

$$H := \left\{ (\sigma_1, \ldots, \sigma_n) \in (\mathbf{Z}/2)^n : \begin{array}{c} \sigma_i = \sigma_j \text{ if for } i \neq j \text{ there is } p \in \Delta_i \cap \Delta_j \\ \text{such that } \partial^1_p(\alpha_{\Delta_i}) = \partial^1_p(\alpha_{\Delta_j}) \neq 0 \end{array} \right\}$$

Therefore,

$$\mathrm{Br}(Y)[2] \simeq H/(1, \ldots, 1)\mathbf{Z}/2$$

Recently a more general version of this theorem has been proved in [2], yelding a similar formula for the Brauer group of conic bundles over 3-folds (that is, with base X of dimension 3). We recall here only a geometric corollary, which is useful to detect stable irrationality of such conic bundles.

Theorem 8.3. *Let k be an algebraically closed field such that $\mathrm{char}\, k \neq 2$ and let $\pi : Y \longrightarrow X$ be a conic bundle over a smooth projective 3-fold X such that $\mathrm{Br}(X)[2] = 0$. Suppose that the discriminant locus Δ of π is a union $\Delta = \Delta_1 \cup \ldots \cup \Delta_n$ of irreducible surfaces Δ_i, with the following conditions:*

(1) *Δ is not irreducible, namely $n \geq 2$;*
(2) *through any irreducible curve of X there pass at most two surfaces amongst the Δ_i's;*
(3) *through any point of B there pass at most three surfaces amongst the Δ_i's;*
(4) *for all $i \neq j$, both Δ_i and Δ_j are locally factorial at every point of $\Delta_i \cap \Delta_j$.*

Now suppose also that

(5) *a general fibre of π above Δ_i consists of two distinct lines and the induced double cover of Δ_i is non-split for each i;*
(6) *for each irreducible component $C_i \subseteq \Delta_i \cap \Delta_j$, the fibres of π above C_i still consist of two distinct lines, but the induced double cover of C_i is split (inside Δ_i or Δ_j) for each $i \neq j$.*

Therefore, $\mathrm{Br}(Y)[2] \neq 0$.

The above theorem has been employed, in the same paper, to determine suitable classes of conic bundle which might be good candidates for the specialisation method. Note, indeed, that adjusting the parameters accordingly in the construction of a conic bundle π can lead to a non-trivial Brauer group and, hence, to a potentially irratioanl variety. In the paper, the strategy is applied with details to a divisor of bi-degree $(2,2)$ in $\mathbf{P}^2 \times \mathbf{P}^3$, treated as conic bundle over $\mathbf{P}^3$. The same example had been studied in ([7]), but from the point of view of a *quadric fibration* over $\mathbf{P}^2$.

8.1. **Reducibility of discriminant.** As one can see in Theorem 8.3, the role of the discriminant locus is particularly significant to determine the right setting for the conic bundle to be stably irrational. A construction to which these ideas may be applied is the case of *graded free type* conic bundle. These are defined as following. Fix a triple $(d_1, d_2, d_3) \in \mathbf{N}^3$ such that $d_i \equiv d_j \mod 2$ and consider a symmetric matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

such that $a_{ij} \in k[X_0, X_1, X_2, X_3]$ are homogeneous polynomial with

$$\deg(a_{ii}) = d_i,\ \deg(a_{ij}) = \frac{d_i + d_j}{2}$$

Define $d := d_1 + d_2 + d_3$ and

$$r_i = \frac{d - d_i}{2},\ s_i = \frac{d + d_i}{2}$$

Therefore, by setting

$$\mathscr{E} := \mathscr{O}_{\mathbf{P}^3}(-r_1) \oplus \mathscr{O}_{\mathbf{P}^3}(-r_2) \oplus \mathscr{O}_{\mathbf{P}^3}(-r_3)$$

we determine a symmetric map between graded free sheaves:

$$\psi : \mathscr{E}^{\vee}(-d) \longrightarrow \mathscr{E}$$

whose action on the sections is given by multiplication by A. By duality, this defines a quadratic form

$$q : \mathscr{E} \otimes \mathscr{E} \longrightarrow \mathscr{O}(d)$$

and, as we have seen before, an associated conic bundle $Y = \{q = 0\} \subseteq \mathbf{P}(\mathscr{E})$ over $\mathbf{P}^3$, whose discriminant locus is the set

$$\Delta = \{p \in \mathbf{P}^3 \mid \det A(p) = 0\}$$

In order to apply the above theorem, one needs to check that Δ satisfies some strict reducibility and intersection conditions. In the forementioned paper [2], these are simplifies employing techniques from the theory of contact of surfaces. Recall that two surfaces $S_1, S_2 \subseteq \mathbf{P}^3$ are said to *have contact* (of order 1) if $S_1 \cap S_2$ is schematically a curve, such that each component has intersection multiplicity along S_1, S_2 at least 2 and there is one of them whose multiplicity is exactly 2.

More specifically, in the above setting, let

$$B := \begin{pmatrix} b_1 & c \\ c & b_2 \end{pmatrix}, \quad N = \begin{pmatrix} c^2 a_{11} - b_1 \det A & ca_{12} & ca_{13} \\ ca_{12} & a_{22} & a_{23} \\ ca_{13} & a_{23} & a_{33} \end{pmatrix}$$

be matrices of homogeneous forms over $\mathbf{P}^3$. If

$$b_2 = \det \begin{pmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{pmatrix}$$

then $\det N = -(\det A)(\det B)$. Here one can prove that b_2 defines a surface which is contact to both $\det A$ and $\det B$. The power of this construction resides in the fact that N defines a conic bundle over $\mathbf{P}^3$, whose restriction to Δ is birationally equivalent to the one defined by A. Thus, in order to apply the Theorem, 8.3, one needs to check that:

- **(a):** $\det A$ and $\det B$ define the irreducible components of $\det N$, namely they define irreducible surfaces in $\mathbf{P}^3$. This guarantees the condition (1) of the Theorem;
- **(b):** $\det A$ and $\det B$ are smooth on the intersection curve $D = \{\det A = \det B = 0\}$. This verifies condition (4);
- **(c):** the double covers induced by N over $\det A$ and $\det B$ are non-trivial. This is condition (5);
- **(d):** N has generically rank 2 on each component of D. This is ensures the first part of condition (6);
- **(e):** the double cover induced by N over the intersection curve D is trivial. This guarantees the last part of condition (6).

Note that (e) is a closed condition and it is the hardest part of the above. In [2], this condition is translated in a more concrete language, using techniques of classical algebraic geometry.

8.2. **Gushel-Mukai varieties.** This class of 4-folds has been recently studied with great interest by Olivier Debarre and Alexander Kuznetsov ([10]). One striking open problem in Gushel-Mukai varieties is whether they are rational or not. Part of this interest comes from the conjectural similarity of the behaviour of these variety and those of cubic 4-folds. Since now, we will work over the field of complex numbers $\mathbf{C}$.

Definition 8.4. Let W be a five dimensional vector space. A *Gushel-Mukai four-fold* X is a smooth dimensionally-transverse intersection

$$X := Q \cap \operatorname{Grass}(2, W) \cap H$$

where $\operatorname{Grass}(2, W) \subseteq \mathbf{P}(\bigwedge^2 W)$ is the Grassmannian, Q is a quadric and H is a hyperplane in $\mathbf{P}(\bigwedge^2 W)$.

One may ask whether the very general Gushel-Mukai 4-fold is stably rational or not and then one may try to apply the specialisation method. This is particularly tempting because these varieties are birational to a certain class of conic bundles over $\mathbf{P}^3$, so one might expect to employ the techniques from the above paragraphs. This is indeed the aim of the very recent paper ([4]) by Christian Böhning and Hans-Christian Graf von Bothmer.

Let us recall the following definitions.

Definition 8.5. Let V be a four dimensional vector space and consider the sheaf $\Omega^1_{\mathbf{P}^3}(2)$ over $\mathbf{P}(V) \simeq \mathbf{P}^3$. A *Gushel-Mukai vector bundle* $\mathscr{E}_\sigma$ is the cokernel of a nowhere vanishing section $\sigma \in H^0(\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(1) \oplus \Omega^1_{\mathbf{P}^3}(2))$, or equivalently an injective morphism

$$\sigma : \mathscr{O}_{\mathbf{P}^3} \longrightarrow \mathscr{O}_{\mathbf{P}^3}(1) \oplus \Omega^1_{\mathbf{P}^3}(2)$$

A *null-correlation bundle* $\mathscr{N}$ is the cokernel of a nowhere vanishing section $\tau \in H^0(\mathbf{P}^3, \Omega^1_{\mathbf{P}^3}(1))$, or equivalently an injective morphism

$$\sigma : \mathscr{O}_{\mathbf{P}^3} \longrightarrow \Omega^1_{\mathbf{P}^3}(1)$$

Then one has the following description.

Proposition 8.6. *A general Gushel-Mukai* 4*-fold X is birational to a conic bundle $\pi : Y_{\varphi,\sigma} \subseteq \mathbf{P}(\mathscr{E}_\sigma^\vee) \longrightarrow \mathbf{P}^3$ associated to a symmetric map*

$$\varphi : \mathscr{E}_\sigma^\vee \longrightarrow \mathscr{E}_\sigma$$

for some Gushel-Mukai vector bundle $\mathscr{E}_\sigma$. This will be called a Gushel-Mukai conic bundle.

The first problem addressed in the article is to decide whether Theorem 8.3 can be applied to this particular class of variety. In order to do that, we have to look for Gushel-Mukai conic bundles in which the discriminant locus Δ breaks up in a suitable way. Now, it is known that Δ is, in general, a so-called Eisenbud-Popesco sextic nodal surface. Thus, we need to determine Gushel-Mukai conic bundles in which Δ breaks up into two cubic surfaces Δ_1, Δ_2; this also guarantees many hypotheses of Theorem 8.3 are satisfied. It turns out that the only possible candidates for Δ_1 and Δ_2 are the so called *del Pezzo cubic symmetroids*, namely: the Cayley cubic with four A_1 singularities, a cubic with two A_1 and one A_3 singularities and a cubic with one A_1 and one A_5 singularitiy.

Moreove, recall the following definition.

Definition 8.7. A *Kummer surface* is a quartic surface in $\mathbf{P}^3$ having exactly 16 nodal singularities.

These surfaces come into the picture because of the following terminology.

Definition 8.8. A *tame degeneration* of a Gushel-Mukai 4-fold consists of

(1) a Gushel-Mukai vector bundle $\mathscr{E}_\sigma$ on $\mathbf{P}^3$;
(2) a symmetric map $\varphi : \mathscr{E}_\sigma^\vee \longrightarrow \mathscr{E}_\sigma$ yelding a conic bundle $\pi : Y_{\varphi,\sigma} \subseteq \mathbf{P}(\mathscr{E}_\sigma^\vee) \longrightarrow \mathbf{P}^3$ with the following properties:
   (a) the discriminant locus Δ of π splits as $\Delta = \Delta_1 \cup \Delta_2$ with Δ_i a del Pezzo cubic symmetroid for $i = 1, 2$;
   (b) each of Δ_i is smooth along $\Delta_1 \cap \Delta_2$;

(c) let τ be the composition

$$\mathscr{O}_{\mathbf{P}^3} \xrightarrow{\sigma} \mathscr{O}_{\mathbf{P}^3}(1) \oplus \Omega^1_{\mathbf{P}^3}(2) \xrightarrow{\mathrm{pr}} \Omega^1_{\mathbf{P}^1}(2)$$

Then τ does not vanish and defines a null-correlation bundle $\mathscr{N}_\sigma$ together with a symmetrix map $\psi : \mathscr{N}_\sigma^\vee \longrightarrow \mathscr{N}_\sigma$ sitting in a diagram

$$\begin{tikzcd} \mathscr{E}_\sigma^\vee \arrow[r, "\varphi"] & \mathscr{E}_\sigma \arrow[d] \\ \mathscr{N}_\sigma^\vee \arrow[u] \arrow[r, "\psi"] & \mathscr{N}_\sigma \end{tikzcd}$$

such that the degeneracy locus of ψ (called the *associated null-correlation quartic)* is a Kummer surface wich is contact to both Δ_1 and Δ_2.

This is the most natural kind of degenerations to look for if one seeks to apply Theorem 8.3. However, in the paper it is proved that this argument fails.

Theorem 8.9. *There is no tame degeneration of Gushel-Mukai fourfolds.*

Still the Theorem could be applicable with respect to a “wilder” class of degenerations. For instance, one might want to remove the condition that the null-correlation quartic is a Kummer surface. This kind of generalised degenerations, however, has neither been found in any computer algebra experiment, so it is still unclear how to proceed in this direction.

Another possibility could be allowing the bundles $\mathscr{E}_\sigma$ and $\mathscr{N}_\sigma$ to degenerate to some sheaves (namely, losing the freeness at some points). Even this possibility has yet to be explored.

8.3. **Conic fibrations in positive characteristic.** An interesting problem which has very lately risen our attention is the behaviour of conic bundles in positive characteristic. This is a meaningful problem, since the specialisation method does not put any restriction on the characteristic of the fields involved. More precisely, we would aim to work in an unequal characteristic setting, namely define a flat family

$$\mathfrak{X} \longrightarrow \mathrm{Spec}(R)$$

of varieties, where R is a discrete valuation ring with residue field k such that $\operatorname{char} k = 2$ and with fraction field K such that $\operatorname{char} K = 0$. Clearly, the special fibre $\mathfrak{X}_0$ is defined over k and the generic fibre is defined over K. This is opposed to the *equal characteristic* context we have treated so far, namely when $\operatorname{char} k = \operatorname{char} K = 0$.

By specialisation, if we manage to prove that

- the special fibre $\mathfrak{X}_0$ has an universally trivial CH_0 resolution of singularities $\widetilde{\mathfrak{X}}_0 \longrightarrow \mathfrak{X}_0$;
- $\widetilde{\mathfrak{X}}_0$ does not have universally trivial CH_0 group;

then we have shown that the geometric generic fibre $\overline{\mathfrak{X}}_\xi := \mathfrak{X}_\xi \times_K \mathrm{Spec}(\overline{K})$ does not have universally trivial CH_0 group and, *a fortiori ratione*, is not stably rational.

This setting becomes slightly more concrete if we assume that $\mathfrak{X}_0$ has a conic bundle structure over some projective space, say $\pi : X = \mathfrak{X}_0 \longrightarrow \mathbf{P}^2_k$. In this case, the most difficult problem to overcome would be determining good invariants which are able to obstruct universal triviality of CH_0 group; one would like to employ the Brauer group and the unramified cohomology as in the equal characteristic case but this poses some notable problems. A roughly hypothetical "path" to follow could be the following.

(1) Firstly, one needs to understand if the non-triviality of Brauer group $\mathrm{Br}(X)[2]$ still represents a good obstruction for universal triviality.
(2) Secondly, it is necessary to re-interpretate the meaning of the residue maps, using Artin-Schreier theory instead of Kummer theory in the Galois cohomology groups.
(3) Thirdly, it is necessary to understand if unramified cohomology can still be used to calculate the above group (in other words, one needs an analogue of Theorem 6.2 when m is not prime to $\mathrm{char}\, k$).
(4) Finally, perhaps the toughest part is to understand the presence of 2-torsion classes in $\mathrm{Br}(X)[2]$ by means of the discriminant profile of the conic bundle (namely, one seeks for an analogue of the sequence in Proposition 6.4).

All of the above is still a matter of speculation and we are currently trying to understand the problem in its best formulation possible.

References

[1] M. Artin – D. Mumford, *Some elementary examples of unirational varieties which are not rational,* Proc. London Math. Soc., **25**, 1972.

[2] A. Auel – Ch. Böhning – H.-Ch. Graf von Bothmer – A. Pirutka, *Conic bundles over threefolds with nontrivial unramified Brauer group*, preprint, 2016, `arXiv : 1610.04995v2`.

[3] A. Beauville, *Variétés de Prym et jacobiennes intermédiaires*, Ann. Sc. Éc. Norm. Sup., **10**, 1977.

[4] Ch. Böhning – H.-Ch. Graf von Bothmer, *Gushel-Mukai fourfolds, vith a view towards irrationality proofs*, preprint, 2017, `arXiv : 1704.01807v1.`

[5] C. H. Clemens, *Double solids*, Adv. of Math., **47**, 1983

[6] J.-L. Colliot-Thélène, *Birational invariants, purity and the Gersten conjecture*, Lect. AMS Summ. School – S. Barbara, 1992.

[7] J.-L. Colliot-Thélène – A. Pirutka, *Hypersurfaces quartiques de dimension 3: non rationalité stable*, Ann. Sc. Éc. Norm. Sup., **49**, 2016.

[8] A. Conte – J. P. Murre, *On a theorem of Morin on the unirationality of the quartic fivefold,* Acc. Sc. Torino, Atti Sc. Fis., **132**, 1998.

[9] A. Corti – J, Kollár – K. E. Smith, *Rational and nearly rational varieties*, Cambridge Univ. Press, 2004.

[10] O. Debarre – A. Kuznetsov, *Gushel-Mukai varieties: classfication and birationalities*, preprint, 2015, `arXiv : 1510.05448v2`.

[11] Ph. Gille – T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Univ. Press, 2006.

[12] Ph. Griffiths – J. Harris, *Principles of algebraic geometry*, Wiley, 1978.

[13] A. Grothendieck, *Le groupe de Brauer*, voll. I and II, Sém. Bourbaki, **9**, 1964 – 1966.

[14] U. Morin, *Sulla unirazionalità delle ipersuperficie algebriche del quarto ordine,* Atti Acc. Naz. Lincei, **24**, 1936.

[15] U. Morin, *Sulla unirazionalità dell'ipersuperficie algebrica del quarto ordine dell'*S_6, Rend. Sem. Mat. Univers. Padova, **21**, 1952.

[16] A. Pirutka, *Varieties that are not stably rational, zero-cycles and unramified cohomology*, Proc. AMS Alg. Geom. Summer Inst., to appear.

[17] B. Segre, *Variazione continua e omotopia in geometria algebrica*, Ann. Mat. pura e appl., **50**, 1960.

[18] C. Voisin, *On the universal* CH_0 *group of cubic hypersurfaces*, to appear in JEMS.

[19] C. Voisin, *Stable birational invariants and the Lüroth problem*, Surveys in Diff. Geom. XXI, International Press, 2016.

[20] C. Voisin, *Stable rationality is not deformation invariant*, 2013.

[21] C. Voisin, *Unirational threefolds with no universal codimension 2 cycles*, Invent. Math., **201**, 2015.