MA371
The Qualitative Theory of Ordinary Differential Equations

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Contents

Warning: I cannot guarantee that these notes are accurate or complete.

Note: Definitions will be in **Bold Text**.

Chapter 1

Introduction

We will study Ordinary Differential Equations, or ODEs, of the form $\dot{\underline{x}} = f(\underline{x})$. The **State Variables** $\underline{x}$ vary in the the **State Space** (or phase space) X. In this course X will be finite dimensional (Partial Differential Equations live in infinite dimensional spaces). X is a metric space, usually $\mathbb{R}^n$. $\underline{x}(t)$ is the **State** of the system $\underline{x} \in X$ at time t. $t \in \mathbb{R}$ is the **Independent Variable**. $f : X \to X$ is a vector field. $f(x)$ gives the speed and direction of the motion at $\underline{x} \in X$. The graph of $(\underline{x}(t), t)$ in X is the **Phase Portrait**. The graph of $\underline{x}(t)$ is called the **Solution**, the trajectory, the orbit of the system or the flow.[1]

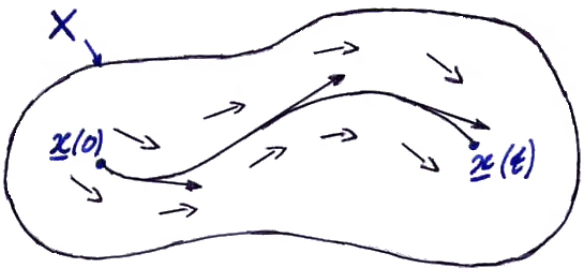

Figure 1.1: A phase portrait.

$\dot{x} = f(x)$ gives an autonomous DE, i.e. one not depending on t. This means that given any point $(x_1, \ldots, x_n) \in X$, $f(x_1, \ldots, x_n)$ will take the same value $\forall\, t \in \mathbb{R}$. This course is concerned with the study of first order autonomous ODEs. Qualitative Theory helps one to understand the local and global behaviour of an ODE without actually having to find explicit solutions to them; thus the subject is of great importance as many ODEs have no explicit solutions.

Example 1.0.1: Take $X = \mathbb{R}$, $\dot{x} = x$ has solution $x(t) = x_0 e^t$, with $x(0) = x_0$. Looking at the phase portrait, one can see that 0 is an unstable fixed point, which is also a global repeller and all orbits not starting at 0 go to $\pm\infty$.

This is the qualitative description of the the behaviour of the system. Thus, if we know f or its graph, we don't need a solution to understand the behaviour of the system. This works for lots of "nice" fs in $\mathbb{R}$, but this becomes very difficult in $\mathbb{R}^{n \geq 3}$.

Example 1.0.2: The below cubic-like function f displays the behaviour one would expect as the ODE $\dot{x} = f(x)$, and is similar to the type of thing studied in first year ODEs.

Example 1.0.3: Consider $\dot{x} = f(x) = \mathrm{Sign}(x)x^2$. $f(x) = 0$ at $x = 0$. For $x > 0$, $x(t) = \frac{1}{x_0^{-1} - t}$. The solution goes to infinity as $t \to \frac{1}{x_0}$. This is called Finite Time Blowup (FTB), i.e. $x \to \infty$ as $t \to T < \infty$.

This happens but we do not worry about it. It doesn't change the phase portrait. However, we do worry about the existence and uniqueness of solutions.

[1] From now on the vector x will not be underlined unless we have something like $\underline{x} = (x, y)$.

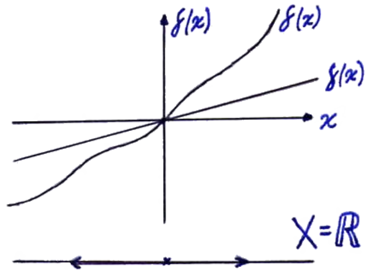

Figure 1.2: Two "nice" functions giving the same result, but only one has an explicit solution.

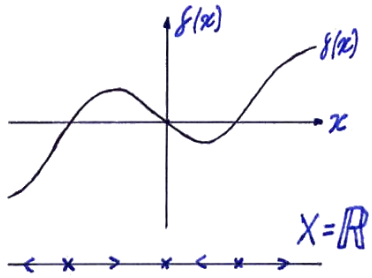

Figure 1.3: Example 1.0.2: a cubic-like function giving two "unstable" fixed points and one "stable" fixed point, as per MA133 Differential Equations.

Example 1.0.4: Consider $\dot{x} = |x|^{\frac{1}{2}}\,\mathrm{Sign}(x)$. We find that for any $T \geq 0$, we have a solution

$$x(t) = \begin{cases} x = 0 & t \in [0, T] \\ x = \frac{1}{4}(t-T)^2 & t \in [T, \infty) \end{cases}$$

so the solution is not unique - what about 0 and T?

Example 1.0.5: With $X = \mathbb{R}^2$. Consider 1-dimensional Simple Harmonic Motion (SHM), given by $\ddot{x} = -x$. Write $y = \dot{x}$ to get $\dot{y} = -x$, $\dot{x} = y$, which is a first order ODE in $\mathbb{R}^2$. We can write this in the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

We can find solutions of the form

$$x(t) = r_0 \cos(t + \phi), \;\; y(t) = r_0 \sin(t + \phi)$$

The phase portrait shows that all solutions are periodic orbits around a fixed point $(0, 0)$, which is structurally unstable.

Aside: One method of solving these is a change of coordinates. In $X = \mathbb{R}^2$ where $\underline{\dot{x}} = A\underline{x}$ with complex eigenvalues for A, write $x = r\cos(\theta)$, $y = r\sin(\theta)$. Then try to get to the form $\dot{r} = 0$, $\dot{\theta} = 1$.

Example 1.0.6: In $X = \mathbb{R}^2$,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

gives $x \to \pm\infty$ as $t \to \infty$ and $y \to 0$ as $t \to \infty$. So $(0, 0)$ is a fixed point. but suppose we had some non-linear, i.e. higher order terms as well? Then the equations would look like

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \mathrm{h.o.t.}$$

This usually can't be solved, but the qualitative question is "what does the phase portrait look like near $(0, 0)$ now?". We will show that it is the same under certain conditions.

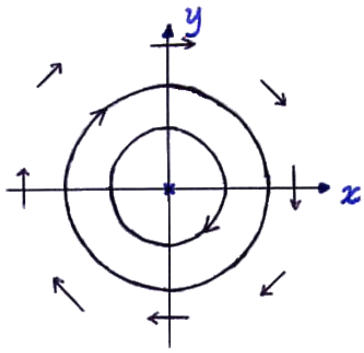

Figure 1.4: Example 1.0.5: the phase portrait of Simple Harmonic Motion.

Example 1.0.7: With $X = \mathbb{R}^3$.

$$\begin{aligned}
\dot{x} &= 10(y - x) \\
\dot{y} &= 28x - y - xz \\
\dot{z} &= \frac{8}{3}z - xy
\end{aligned}$$

The Lorenz equations have only two quadratic but terms, but the phase portrait is chaotic with a strange attractor:

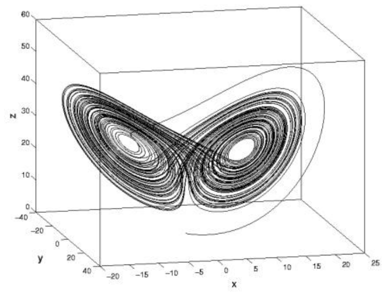

Figure 1.5: The well-known Lorenz attractor.

We are interested in behaviour (such as periodicity, going to a limit or infinity, etc) which is invariant under "nice" changes of coordinates. By a "nice" change of coordinates h with $\underline{y} = h(\underline{x})$ we mean a diffeomorphism, i.e. a differentiable bijection with differentiable inverse. We are also interested in properties invariant under parameterisation of time, e.g. new time $s = \alpha(t)$, where $\alpha : \mathbb{R} \to \mathbb{R}$.

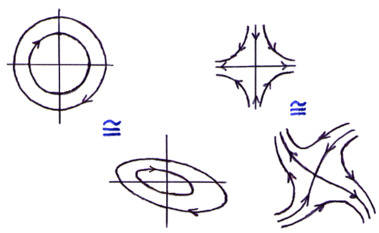

Figure 1.6: These systems have been subjected to a diffeomorphism, but their phase portraits display the same sort of behaviors.

What does it mean to say that there does not exist explicit solutions? Consider $\dot{y} = y^2 - t$ - this has no explicit solution, but we could define a new function $L(y_0, t)$ as being the solution, but obviously this is not helpful. Similarly the pendulum solution $\ddot{\theta} = -\sin(\theta)$ can be solved using elliptic functions, but these are not "normal".

Chapter 2

One Dimensional ODES ($X = \mathbb{R}^1$)

2.1 Flows, Existence and Uniqueness

It is more natural to write "solutions" as functions of starting position and time.

Definition: $\phi : X \times \mathbb{R} \to \mathbb{R}$ is called the **Flow**; $\phi(x_0, t) = x(t)$ the place one gets to starting at x_0 at $t = 0$ and solving until time t.

We need ϕ to have some obvious properties:

(I) $\phi(x_0, 0) = x_0$

(II) $\phi(\phi(x_0, s), t) = \phi(x_0, s + t)$

A function ϕ satisfying these properties is a candidate for the solution of the ODE.[1] To solve $\dot{x} = f(x)$, then it should satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\phi(x, t)\right) = f(\phi(x, t))$$

Equivalently, integrate both sides to get

$$\phi(x_0, t) = x_0 + \int_0^t f\left(\phi(x_0, s)\right) \mathrm{d}s$$

So, given f, does ϕ exist and is it unique?

Definition: A function $f : X \to Y$ metric spaces is **Lipschitz** with Lipschitz constant L if

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

for some $L \in \mathbb{R}$, and for all $x, y \in X$.

Example 2.1.1: If $f : \mathbb{R} \to \mathbb{R}$, f is Lipschitz with constant L, the function remains in the region bounded by lines of slope $\pm L$ through the point $(x, f(x))$.

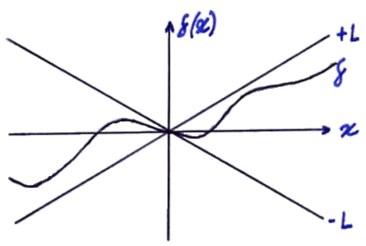

Figure 2.1: Example 2.1.1: a Lipschitz function. f cannot escape the lines of slope $\pm L$.

[1] If it does so for $s, t \geq 0$ then it is called a semi flow in some books.

Note: Lipschitz $\Rightarrow$ Continuous, but Lipschitz $\not\Rightarrow$ Differentiable.

Definition: f is **Locally Lipschitz** at the point x^* with constant L if

$$\|f(x^*) - f(y)\| \leq L\|x^* - y\|$$

for all y in a neighborhood of x^*.

Example 2.1.2: $f(x) = |x|^{\frac{1}{2}}$ is not locally Lipschitz at 0 because L would have to be $\pm\infty$.

Outline of what we will do next:

(1) Theorem: f Lipschitz $\Rightarrow \exists!\, \phi$ solving $\dot{x} = f(x)$.

(2) Worry: How long does ϕ exist for? The theorem is only local.

(3) Proof: An algorithm to find a sequence of functions converging to ϕ.

How to solve $\dot{x} = f(x)$?[2] We shall try to approximate the solution with a sequence of functions $\{u_i\}$:

$$\begin{aligned}
u_0(x_0, t) &= x_0 \\
u_1(x_0, t) &= x_0 + \int_0^t f\left(u_0(x_0, s)\right) \mathrm{d}s \\
&\vdots \\
u_{n+1}(x_0, t) &= x_0 + \int_0^t f\left(u_n(x_0, s)\right) \mathrm{d}s
\end{aligned}$$

Note that if the sequence of functions converges to ϕ, then

$$\phi(x_0, t) = x_0 + \int_0^t f\left(\phi(x_0, s)\right) \mathrm{d}s$$

Implying that ϕ is a flow solving the equation.

Definition: This is called **Picard Iteration**, and we denote it by $u_{n+1} = P(u_n)$.

Example 2.1.3: Let us check that for $\dot{x} = x$ the Picard Iteration gives $x(t) = x_0 e^t$:

$$\begin{aligned}
u_0(x_0, t) &= x_0 \\
u_1(x_0, t) &= x_0 + x_0 t \\
u_2(x_0, t) &= x_0 \left(1 + t + \frac{t^2}{2}\right) \\
&\vdots \\
u_n = (x_0, t) &= x_0 \left(1 + t + \frac{t^2}{2} + \ldots \frac{t^n}{n!}\right)
\end{aligned}$$

So $u_n(x_0, t) \to x_0 e^t$ as $n \to \infty$.

The proof of the theorem relies on the fact that Picard Iteration is a contraction mapping on an appropriate function space:

$$\|P(u) - P(v)\| \leq K\|u - v\|$$

for some fixed $K \in [0, 1)$ (see MA222 Metric Spaces or MA225 Differentiation for the contraction mapping theorem). We use the fact that f is Lipschitz when proving that P is a contraction. Do this as an exercise.

Note: The worry, (2), arises since the space of functions we consider is only defined in some interval $|t| \leq \tau$ for some constant τ.

[2] See Grimshaw for details.

Theorem: If f is locally Lipschitz then $\forall y \in X \, \exists \varepsilon, \tau > 0$ such that solutions exist and are unique for each $|x - y| < \varepsilon$ and $t \in [-\tau, \tau]$. Furthermore, $\phi : X \times \mathbb{R} \mapsto X$ is Lipschitz.

Can we use the theorem to extend the solutions by gluing end-pieces together? Pick $x_1 \in X$ and define $x_1, x_2, \ldots \in X$ by applying the theorem - a solution through x_i exists in the interval $[-\tau_i, \tau_i]$, so define $x_{i+1} = \phi(x_1, \tau_i)$. What can happen?

- If there is a fixed point x^* then $x_i \to x^*$ as $t \to \infty$
- $\sum_i \tau_i < \infty$ but $x_i \to \pm\infty$ (FTB)

What else?

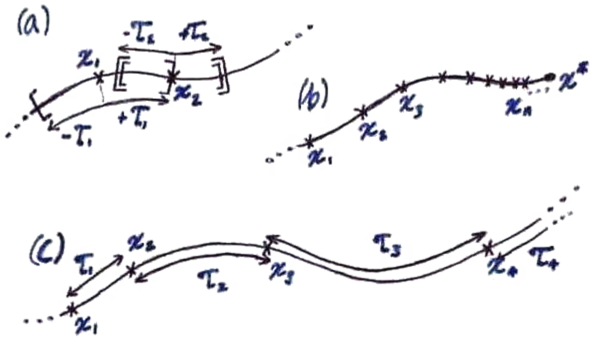

Figure 2.2: (a) "Gluing" pieces together. (b) A fixed Point. (c) Finite Time Blowup.

Theorem (Extension Theorem)**:** Let f be locally Lipschitz. Then for all $x_0 \in X$ there is a maximal time interval $[\alpha, \beta]$, $\alpha < 0 < \beta$ where either or both could be infinite, on which there is a unique solution. If $\beta < \infty$, then $\phi(x,t) \to \infty$ as $t \to \beta$, i.e. we get FTB.

Corollary: If $\phi(x,t)$ remains in a compact set, then $\beta = \infty$ and there is a unique solution for all $t > 0$.

Prove these as an exercise.

2.2 Orbits

Definition: $\mathrm{O}(x)$ - the **Orbit** of x - is the set $\{\phi(x,t) : t \in [\alpha, \beta]\}$. $\mathrm{O}^+(x)$ is the positive orbit of x, defined on $t \in (0, \beta]$.

Definition: x^* is a **Fixed Point** if $\mathrm{O}(x^*) = x^*$, or equivalently $f(x^*) = 0$.

Definition: We say that x is a **Periodic Point** with period $T > 0$ if $\phi(x,T) = x$ but $\phi(x,t) \neq x$ for $t \in (0,T)$. $\mathrm{O}(x)$ is a **Periodic Orbit** of period T.

Definition: If x^* is a fixed point and $\phi(x,t) \to x^*$ as $t \to \pm\infty$, then $\mathrm{O}(x)$ is a **Homoclinic Orbit**.

Definition: If x^*, y^* are distinct fixed points, and $x^* \leftarrow \phi(x,t) \to y^*$ as $-\infty \leftarrow t \to \infty$, then $\mathrm{O}(x)$ is a **Heteroclinic Orbit**.

Definition: A **Heteroclinic Loop** is a union of heteroclinic orbits joining fixed points $x_1^*, \ldots, x_n^*, x_1^*$ cyclically.

For more exotic examples, consider the Torus in $\mathbb{R}^3$ with the DE defined on the surface as $\dot{\theta}_1 = \omega_1$, $\dot{\theta}_2 = \omega_2$. If $\frac{\omega_1}{\omega_2} \in \mathbb{Q}$ then the orbits are periodic - for every ω_1 times you go around one way, you go around the other way ω_2 times. But if $\frac{\omega_1}{\omega_1} \notin \mathbb{Q}$, then any orbit fills the surface of the torus densely. There are other more complicated geometric objects defined by the flow, such as strange attractors. This happens in $\mathbb{R}^{n \geq 3}$. Most of the interesting objects in a flow can be characterised by one or more of these properties:

(I) Invariance

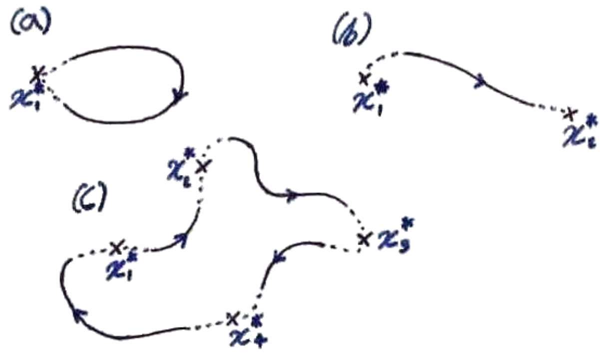

Figure 2.3: (a) A homoclinic orbit. (b) A heteroclinic orbit. (c) A heteroclinic cycle.

(II) ω-limit sets

(III) Stability

Definition: A set Λ is **Invariant Under** f if for all $x \in \Lambda$, $\phi(x,t) \in \Lambda$ for every $t \in \mathbb{R}$ and **Forward Invariant** if only satisfied for all $t \geq 0$.

Examples of invariant sets include fixed points, periodic orbits, the whole space X, and the orbits of every $x \in X$. Invariance is important, but clearly far too general. We have to capture some notion of the eventual behaviour of the flow. We could look at points $y \in X$ such that $\phi(x,t) \to y$ as $t \to \infty$, but this is too narrow - points in a periodic orbit cannot satisfy this for example.

Definition: Given $x \in X$, the ω**-limit set** of x, $\omega(x)$, is the set $\{y \in X : \phi(x,t_n) \to y\}$ for some increasing sequence $t_n \to \infty$.

Warning: The last point is important, otherwise any point $\phi(x,T)$ is in $\omega(x)$ for all $T > 0$.

Example 2.2.1: Some examples:

(I) If x is a fixed point, then $\omega(x) = x$.

(II) If x is on a periodic orbit, then $\omega(x)$ is the whole periodic orbit.

(III) Consider a saddle point x_0 with a homoclinic orbit attached, and an unstable focus point x_1 inside the homoclinic orbit such that orbits inside spiral out from the centre to the edge: Let

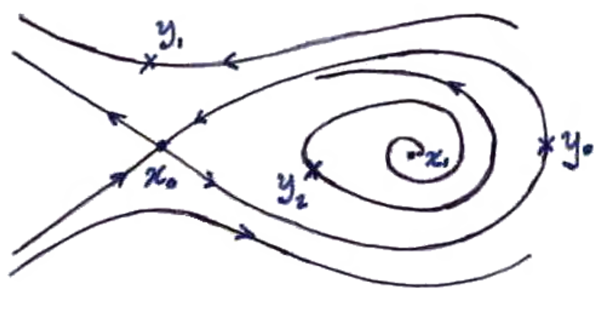

Figure 2.4: Example 2.2.1 (III).

y_0 be a point on the homoclinic orbit, y_1 a point on an orbit outside it, and y_2 a point on a spiral inside it. Then

- $\omega(x_0) = x_0$, $\omega(y_0) = x_0$,
- $\omega(x_1) = x_1$, $\omega(y_1) = \varnothing$, and
- $\omega(y_2) = \{x_0\} \cup \{\text{homoclinic orbit}\}$.

Theorem: If $\mathrm{O}(x)$ is bounded, then $\omega(x)$ is non-empty, closed, connected and invariant.

Proof. (Sketch)

Non-empty: $O(x)$ is bounded, so $O(x) \subseteq K$ is compact. Take any sequence of times $t_n \to \infty$. Then $\phi(x, t_n)$ is a sequence of points in K, so there is a convergent subsequence converging to y say, so $y \in \omega(x)$.

Invariant: Suppose $y \in \omega(x)$. Then there is a sequence $t_n \to \infty$ such that $\phi(x, t_n) \to y$. Consider the point $\phi(y, \tau)$. Then $\phi(x, t_n + \tau) \to \phi(y, \tau)$ by the continuity of ϕ.

Connected: If $\omega(x)$ were disconnected, then it would have at least two disjoint sets A and B such that $A \cup B = \omega(x)$, $A \cap B = \varnothing$. We can find disjoint open neighbourhoods of A and B. $\phi(x, t)$ must keep crossing from one to the other, so we can pick a sequence of times $t_n \to \infty$ such that $\phi(x, t_n)$ is in neither A nor B. We have a sequence of points in a compact set[3], so there must be a subsequence converging to a point of $\omega(x)$, which is a contradiction.

Closed: Suppose we have $z_1, z_2, \ldots \in \omega(x)$ such that $z_n \to y$. So there are $\phi(x, t_{ni})$ such that $t_{ni} \to \infty$ and $\phi(x, t_{ni}) \to z_n$ as $i \to \infty$. Pick from each of these sequences of times a sequence such that the times are increasing, and we get a sequence $\phi(x, t_n)$ such that $t_n \to \infty$ and $\phi(x, t_n) \to y$ as $n \to \infty$. See the handout for more details. ■

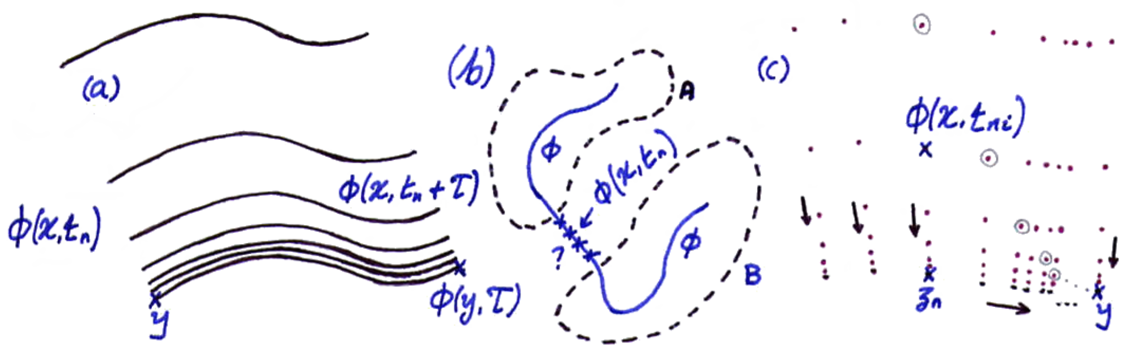

Figure 2.5: (a) Invariance. (b) Connected. (c) Closed: the circled dots represent one possible sequence $\phi(x, t_n)$.

2.3 Stability

Liaponov - "Start near, stay near".. Asymptotic - "Start near, get close to and tend to as $t \to \infty$".

Examples 2.3.1:

- $\dot{x} = y, \dot{y} = -x$. $(0, 0)$ is a fixed point - Liaponov stable but not asymptoticly stable. Any periodic orbit is Liaponov stable but not asymptotically stable.
- $\dot{x} = -x, \dot{y} = -y$. $(0, 0)$ is both Liaponov stable and asymptotically stable.

We will see that Asymptotically stable implies Liaponov stable.

Definition: An invariant set Λ is called **Liaponov Stable** if for all neighbourhoods $V \supseteq \Lambda$ there is a neighbourhood $U \subseteq V$ such that $x \in U \Rightarrow \phi(x, t) \in V$ for all $t > 0$.

Definition: Λ is **Asymptotically Stable** if it is Liaponov Stable and if there is a neighbourhood $V \supseteq \Lambda$ such that $\phi(x, t) \to \Lambda$ for all $x \in V$.

Lemma: If Λ is asymptotically stable, then it is also Liaponov stable.

Example 2.3.1: A fixed point x^* with a homoclinic orbit which is neither Liaponov stable nor asymptotically stable, since we have to go around the loop to get back to it, even though for all $x \in V$, $\phi(x, t) \to x^*$ as $t \to \infty$. See figure 2.6.

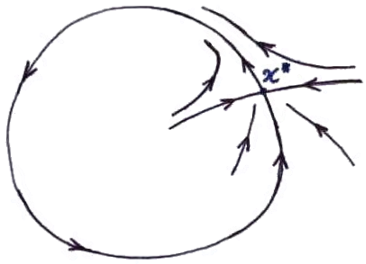

Figure 2.6: Example 2.3.1.

Note: Asymptotically stable sets are called sinks or attractors. There are many different definitions for each of these. Invariant sets asymptotically stable under time reversal are called sources or repeller. We tend not to use the word unstable, since it means not stable, and is thus not the opposite of stable, and we do not wish to confuse things like sources with things like saddles.

Definition: If Λ is invariant and asymptotically stable, then $\mathrm{B}(\Lambda)$, the **Basin of Attraction**, is the set $\{x \in X : \phi(x,t) \to \Lambda\}$ as $t \to \infty$.

Example 2.3.2: An asymptotically stable point inside a homoclinic orbit with everything spiraling towards it. Then the basin of attraction is everything within the homoclinic orbit; see figure 2.7.

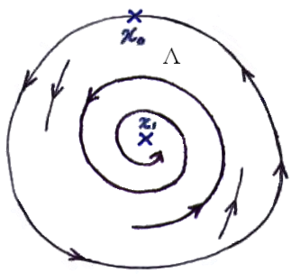

Figure 2.7: Example 2.3.2. Λ is the whole region inside the homoclinic orbit.

Exercise: Show that $\mathrm{B}(\Lambda)$ is always open.

Definition: If Λ is asymptotically stable, invariant, and $\mathrm{B}(\Lambda) = X$, then Λ is a **Global Attractor**.

We can now describe phase spaces with $f : \mathbb{R} \mapsto \mathbb{R}$. Recall the usual results.

Exercise: Show that for $X = \mathbb{R}$, x^* a fixed point, $f'(x^*) < 0 \Rightarrow x^*$ asymptotically stable.

Definition: If f is $\mathcal{C}^1$ and x_0 is a fixed point such that $|f'(x_0)| \neq 0$ then we say that x_0 is a **Hyperbolic** fixed point.

In rough: Hyperbolic points are structurally stable, non hyperbolic points are not. Associated with bifurcations. Hyperbolic stability: slight changes don't change the qualitative picture.

Note: We must be careful by what we mean by "slight change". Stability of a structural kind depends on an appropriate definition of "small change" - for $f \in \mathcal{C}^1$, both f and f' change by a small amount.

[3] If U and V are neighbourhoods of A and B respectively, then $X \setminus (U \cap V)$ is compact.

Chapter 3

Two Dimensional Flows ($X = \mathbb{R}^2$)

3.1 Hamiltonian Flows and Hamiltonians

If a quantity is conserved by a flow, this "reduces the problem by one dimension", where you look at lines where the quantity is constant.

Example 3.1.1: $\dot{x} = y, \dot{y} = x^3 - x$. consider

$$H(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4$$

Compute $\frac{\mathrm{d}H}{\mathrm{d}t}$ along the flow.

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = y(x - x^3) + y(x^3 - x) = 0$$

So H does not vary along orbits of the flow. Look at sets $\{H = c\}$. Draw the fixed points, $y = 0, x = 0, \pm 1$. Calculate H at the fixed points: $H(0, 0) = 0$, $H(\pm 1, 0) = \frac{1}{4}$. Draw the level sets remembering techniques for phase portraits. We know the behaviour of the flow:

(I) At the fixed points x^* are such that $f(x^*) = 0$.

(II) All orbits of the flow must lie within a set of type $\{H = c\}$, i.e. $\frac{\mathrm{d}H}{\mathrm{d}t} = 0$.

(III) For $0 < c < \frac{1}{4}$ the sets look like ellipses. Contains no fixed points anywhere on the curve $\Rightarrow |f(x)|$ not zero anywhere $\Rightarrow |f(x)|$ attains a positive minimum on the curve $\Rightarrow$ velocity is bounded away from 0 on the curve $\Rightarrow$ periodic orbit of flow with finite period T_c.

(IV) $c = \frac{1}{4}$ uses similar arguments; note that the fixed points are in this set. There is a heteroclinic orbit between the fixed points $(1, 0)$ and $(-1, 0)$, so any orbit starting on one of these heteroclinic orbits will tend tend to the appropriate fixed points as $t \to \pm\infty$ (see figure 3.1). Orbits not on the heteroclinic orbits will tend to ∞ or $-\infty$ as $t \to \infty$ or the fixed points as $t \to -\infty$.

Observe that we have learned a lot of information about the flow with solving the equations.

Example 3.1.2: Pendulum: $\ddot{\theta} = -\sin(\theta)$. Write $p = \dot{\theta}$ to get $\dot{\theta} = p$, $\dot{p} = -\sin(\theta)$. Consider $H(p, \theta) = \frac{1}{2}p^2 - \cos(\theta)$.

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial \theta}\dot{\theta} = -p\sin(\theta) + p\sin(\theta) = 0$$

So H is constant on all orbits. Sets $\{H = c\}$. There are fixed points at $(0, 2k\pi)$ and $(0, (2k + 1)\pi)$ where $k \in \mathbb{Z}$, with periodic orbits around $(0, 2k\pi)$ and homoclinic orbits joining $(0, (2k + 1)\pi)$.

Note: We have two possibilities for the phase space: $\mathbb{R}^2$ or $\mathcal{S}^1 \times \mathbb{R}$, the latter being the more natural choice.

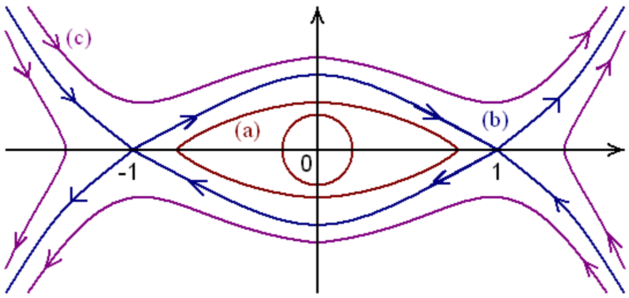

Figure 3.1: Example 3.1.1: The level sets $H = c$ show the fixed points, heteroclinic orbits and periodic orbits, enabling us to get a good idea of the flow. In (a), $c < \frac{1}{4}$ (III), in (b) $c = \frac{1}{4}$ (III) while in (c), $c > \frac{1}{4}$.

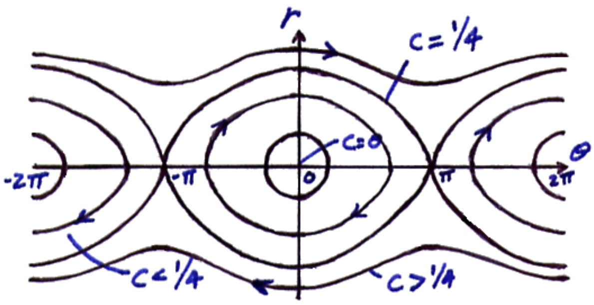

Figure 3.2: Example 3.1.2: The level sets $H = c$ show the fixed points, heteroclinic orbits and periodic orbits, enabling us to get a good idea of the flow. Note that the flow can be taken between 0 and 2π or between $-\pi$ and π.

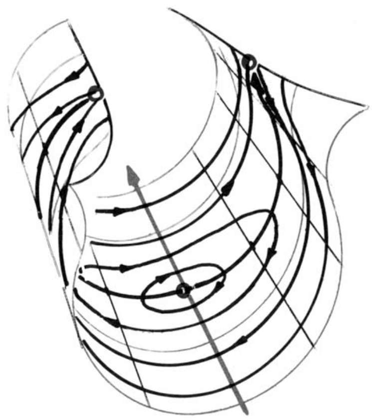

Figure 3.3: The phase space is a cylinder, but it makes life easier to "unwrap" it into $\mathbb{R} \times [0, 2\pi)$.

Definition: A system is **Conservative** (or Hamiltonian) if there is a function $H : X \to \mathbb{R}$ such that $\frac{\mathrm{d}}{\mathrm{d}t}(H(\phi(x,t))) = 0$.

Definition: Such a function H is known as the **Hamiltonian Function**.

In real systems, there is often a conserved quantity, although not necessarily energy. If the system is Hamiltonian, the orbits lie in the level sets of H, locally of dimension one less than that of the phase space X (i.e. of co-dimension 1). In general it is hard to know if H even exists, and is usually difficult to find when it does.

3.2 Liaponov Functions

Example 3.2.1 (1)**:** The damped pendulum has the system $\ddot{\theta} = -k\dot{\theta} - \sin(\theta)$, where $k > 0$. Write $\dot{\theta} = p$, $\dot{p} = -kp - \sin(\theta)$. Consider $V(\theta, p) = \frac{1}{2}p^2 - \cos(\theta)$ - we don't call this function H since it is no longer a Hamiltonian:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial \theta}\dot{\theta} + \frac{\partial V}{\partial p}\dot{p} = -kp^2 \leq 0$$

Thus $\dot{H} = 0 \Leftrightarrow p = 0$. Now we expect that solutions cross the lines $V = c$ in the direction of decreasing c. This is not an ideal example, since there is a line $p = 0$ where $\dot{V} = 0$.

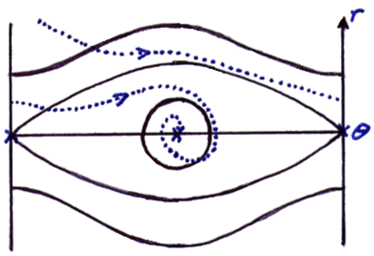

Figure 3.4: The black whole lines are the level sets $H = c$ and the blue dotted lines are the solutions. These two lines always cross at the same angle.

Example 3.2.2 (2)**:** $\dot{x} = x^2 - x - 2xy$, $\dot{y} = y^2 - 2y + 5xy$. There is a fixed point at the origin. The linear part of the system suggests that near near to $(0, 0)$ everything decreases, so the origin is asymptotically stable. Try $V(x, y) = \frac{1}{2}(x^2 + y^2)$:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = x\dot{x} + y\dot{y} = -x^2(1 - x + 2y) - y^2(2 - y + 5x)$$

If $\dot{V} \leq 0$ at least in the region where $x - 2y \leq 1$ and $y - 5x \leq 2$ we should be able to find a ball inside this region where everything goes to the origin. We can't say anything about orbits in the shaded region but not the ball, since they could go out of the region.

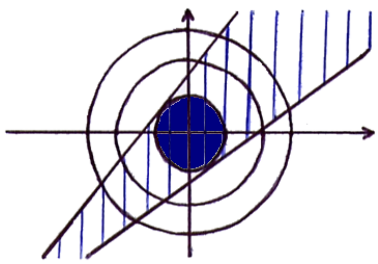

Figure 3.5: Every orbit within the blue ball is asymptotically stable.

Definition: A **Liaponov Function** $V : X \to \mathbb{R}$ for a flow ϕ on X is a function such that $V(\phi(x, t)) \leq V(x) \,\forall\, t > 0$, $x \in X$. Usually we have that $V \in \mathcal{C}^1$ and $\dot{V} \leq 0$.

We're often concerned with local behaviour near to a fixed point.

Definition: Let x_0 be a fixed point. Suppose $V : X \to \mathbb{R}$ is differentiable in $U \setminus \{x_0\}$ where U is an open neighbourhood of x_0 such that

(a) $V(x_0) = 0$ and $V(x) > 0 \,\forall\, x \in U \setminus \{x_0\}$

(b) $\dot{V} \leq 0$ in $U \setminus \{x_0\}$ or

(c) $\dot{V} < 0$ in $U \setminus \{x_0\}$.

If V satisfies (a) and (b) it is called a **Liaponov Function for** x_0. If in addition it satisfies (c) then it is called a **Strict Liaponov Function.**

Note: We can fix $V(x_0) = 0$ by adding a constant to V.

Examples 3.2.1:

- In the first example V was a strict Liaponov function for $(0, 0)$.
- For the damped pendulum (c) was not satisfied since $V = 0$ whenever $p = 0$. But in fact the fixed point is asymptotically stable, but we need more theory.

Theorem (Liaponov's First Theorem)**:** If there is a Liaponov function for a fixed point x_0, then x_0 is Liaponov stable.

Proof. Given U a neighbourhood of x_0 we want to show that there is a neighbourhood $V' \subseteq U$ such that $\forall\, x \in V'$, $\phi(x,t) \in U \,\forall\, t > 0$. Pick $\delta > 0$ small enough such that $B(x_0, \delta) \subsetneq U$. Consider the surface of $B(x_0, \delta)$; it is the surface of the sphere $\mathcal{S}_\delta(x_0)$. V takes positive values at all points on $\mathcal{S}_\delta(x_0)$. Since $\mathcal{S}_\delta(x_0)$ is a compact set there is a minimum value $\alpha > 0$ of V on $\mathcal{S}_\delta(x_0)$.
Define the open set $V' := \{x \in V(x) < \alpha\} \cap B(x_0, \delta)$. Since α was a minimum $V' \subsetneq B(x_0, \delta)$ (points on the boundary have $V \geq \alpha$). Now $\forall\, x \in V'$, $V(x) < \alpha$. so $V(\phi(x,t)) < \alpha$ for $t \geq 0$, since $\dot{V} \leq 0$. So $V(\phi(x,t)) \ngeq \alpha$ for $t \geq 0$. So $\phi(x,t) \notin \mathcal{S}_\delta(x_0) \,\forall\, t \geq 0$. So $\phi(x,t)$ remains inside $B(x_0, \delta) \subseteq U \,\forall\, t \geq 0$. ■

Theorem (Liaponov's Second Theorem)**:** If there is a strict Liaponov function for a fixed point x_0, then x_0 is asymptotically stable.

Proof. Now we have $\dot{V} < 0$ in $U \setminus \{x_0\}$. We already know from the first theorem that x_0 is L-stable. We just need to show that $\phi(x,t) \to x_0$ as $t \to \infty$. Choose sets as before and pick $y \in V'$. Then $\phi(y,t) \in B(x_0, \delta) \,\forall\, t \geq 0$, so the orbit stays bounded (so $\omega(y) \neq \varnothing$). If $\omega(y) = x_0$ then we are done, and $(\phi(y,t) \to \omega(y)$ as $t \to \infty)$.
Suppose there is a $z \in \omega(y)$ such that $z \neq x_0$. So there is a sequence of times $\{t_n\}_{n=1}^{\infty}$ with $t_n \to \infty$ and $\phi(y, t_n) \to z$. Since we have $\dot{V} < 0$, $\phi(y,t)$ is decreasing monotonically to $V(z)(> 0)$. Consider the orbit $\phi(z,t)$ through z. $\dot{V} < 0$ at z, so there is a time τ such that $V(\phi(z,\tau)) < V(z)$. By continuity, for points z' close enough to z, $V(\phi(z',\tau)) < V(z)$. So now just choose $z' = \phi(y, t_n)$ for n large enough. But then $V(\phi(y, t_n + \tau)) < V(z)$, which contradicts $V(\phi(y,t)) \searrow V(z)$ as $t \to \infty$. This is a contradiction, so there is no non-x_0 element of $\omega(y)$. So x_0 is asymptotically stable. ■

Define $E := \{x : \dot{V} = 0\}$ for a Liaponov function V. Define $M_c := \{x : V(\phi(x,t)) = c \,\forall\, t \geq 0\}$. M_c contains entire forward orbits. If $x \in M_c$, then so is $\phi(x,t)$, $f \geq 0$, so $M_c \subseteq E$. In the damped pendulum case, $M_C \subseteq \{p = 0\}$ and consists of entire forward orbits. The only orbits which remain in $\{p = 0\}$ are the two fixed points $(0,0)$ and $(\pi, 0)$. For other points consider that $\dot{p} \neq 0$. The only non-empty sets M_c are $M_1 = (\pi, 0)$ and $M_{-1} = (0, 0)$. $M_{c \neq \pm 1} = \varnothing$.

Theorem (La Salle's Principle)**:** If V is a Liaponov function for a flow ϕ then $\forall\, x \in X$, $\exists\, c$ such that $\omega(x) \subseteq M_c$.

Apply this to the damped pendulum: $\forall x \in X$, $\omega(x) \subseteq M_c$ for some c so $\forall x$, $\omega(x) = (0,0)$ or $(\pi, 0)$. There are some orbits that tend to $(\pi, 0)$ (in fact there are only two), but we already know that $(0,0)$ is Liaponov Stable, so orbits staring near $(0,0)$ don't go to $(\pi, 0) \Rightarrow$ must go to $(0,0) \Rightarrow (0,0)$ is Liaponov Stable.

Proof. Pick $x \in X$. $\dot{V} \leq 0$, or more generally V is not decreasing. Let $c := \inf\{V(\phi(x,t)) : t \geq 0\}$. If $c = -\infty$, $\phi(x,t) \to \infty \Rightarrow \omega(x) = \varnothing$ and there is nothing to prove. Assume $c \neq -\infty$ and that $\omega(x) \neq \varnothing$. Let $y \in \omega(x)$, so there is a sequence of times $\{t_n\}_{n=1}^{\infty}$ tending to infinity such that $\phi(x, t_n) \to y$. By the continuity of V and $V(\phi(x, t_n)) \searrow c$ we have $V(y) = c$. We need to show that $y \in M_c$, so we first need to prove that $V(\phi(y,t)) = c \,\forall\, t \geq 0$. Suppose not. Then there is an s such that $V(\phi(y,s)) < c$. But then consider the sequence of points $\phi(x, t_n + s) \to \phi(y, s) \Rightarrow V(\phi(x, t_n + s)) \to V(\phi(y,s))$. this contradicts the definition of c so if $V(\phi(x,t))$ so $y \in M_c$ and we are done. ■

Exercise: Prove Liaponov's second theorem; from Liaponov's first theorem and La Salle's Principle it should take only a few steps.

The use of "Liaponov Functions" as bounding functions: take a set $K \neq \varnothing$. $\exists$ function such that $\dot{V} < 0$ outside K. Want to identify a set K' such that all orbits eventually enter K' and never leave it.

Theorem: Let $V : \mathbb{R}^n \to \mathbb{R}$ be differentiable and let K be a region outside which $\dot{V} \leq -\delta$, for some $\delta > 0$. Let $\alpha := \sup\{V(x) : x \in \partial K\}$. Then $K' := V_\alpha := \{x : V(x) \leq \alpha\}$ is such that all orbits enter and remain in K'.

The proof is a simple exercise, but finding V is usually awful.

Definition: A system is called a **Gradient System** if $x = -\nabla V(x)$ for some function $V : X \to X$.

For such functions, V is a Liaponov function: $\dot{V} = -(\nabla V)^2 \leq 0$ and $\dot{V} = 0$ only at fixed points. La Salle's invariance principle implies that all orbits tend to fixed points.

3.3 Fixed Points and Nearby Behaviour

Most theorems are in $\mathbb{R}^n$, and most examples are in $\mathbb{R}^2$. Suppose $\dot{x} = f(x)$ has a fixed point at x_0. Change variables to move the fixed point to the origin:

$$y = x - x_0, \ \dot{y} = \dot{x} = f(x) = f(y + x_0)$$

So the new equation $\dot{y} = f(y + x_0) =: g(y)$ has a fixed point at 0: $g(0) = f(x_0) = 0$. So now we have that, without loss of generality, $f(x) = \dot{x}$ and $f(0) = 0$. Our General system can be written in the form $\dot{x} = Ax + \text{h.o.t.}$, where A is an $n \times n$ matrix, and the higher order terms are quadratic or higher. Ax is called the **Linear Part**, h.o.t. the **Non-Linear Part**. We want to

(I) Recall how to solve/understand the behaviour of the linear system $\dot{x} = Ax$.

(II) Understand how the addition of non-linear terms affects the behaviour.

3.3.1 The Linear Part

Consider the hyperbolic case, i.e. where A has no eigenvalues with 0 real part (c.f. in the 1D case $f'(0) \neq 0$); the non-hyperbolic case comes later. Consider a linear change of variables $y = Bx$, where B is non-singular. Then $\dot{y} = B\dot{x} = BAx = BAB^{-1}y$. BAB^{-1} has the same eigenvalues as A, and by choosing B carefully, we can put BAB^{-1} into Jordan Norma Form. Recall that the JNF makes $\dot{x} = Ax$ much easier to solve by allowing us to solve each block separately. In $\mathbb{R}^2$ there are only four possibilities:

$$\text{(A)} \underbrace{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}}_{\lambda_1 \neq \lambda_2}, \text{(B)} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \text{(C)} \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}, \text{(D)} \begin{pmatrix} \lambda_1 & \omega \\ -\omega & \lambda_1 \end{pmatrix}$$

(A) There are three cases:

1. If $\lambda_1, \lambda_2 > 0$, then the origin is a **Repeller** or **Source**. Orbits are repelled at different rates in different directions according to the ratio of the two eigenvalues; see 1^{st} year ODEs.
2. If $\lambda_1 > 0, \lambda_2 < 0$, then the origin is a **Saddle**:
3. If $\lambda_1, \lambda_2 < 0$, then the origin is an **Attractor** or **Sink**.

(B) Gives two cases:

1. If $\lambda_1 < 0$ then the origin is called a **Star Node**, and it attracts all orbits equally.
2. If $\lambda_1 > 0$ then the origin repels all orbits equally in all directions.

(C) In this case we have 2 equal eigenvalues but only one eigenvector; see diagram.

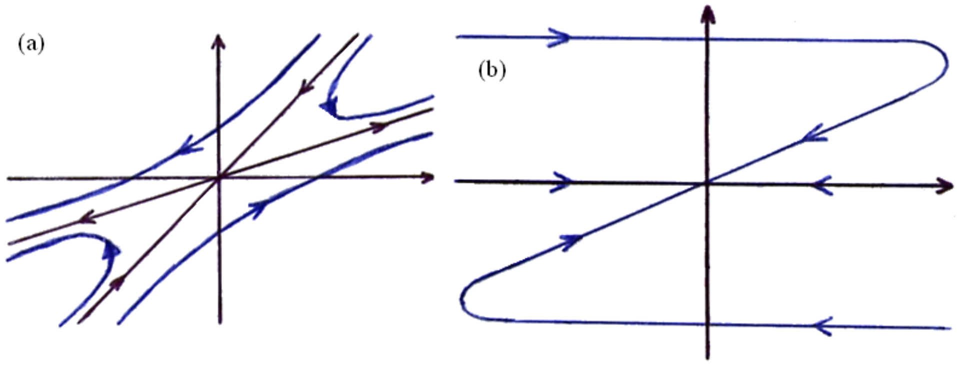

Figure 3.6: (a) A Saddle (b) Case C

(D) We have two cases:

1. If $\lambda_1 > 0$ then orbits spiral outwards clockwise - the origin is an **Unstable Focus**.
2. If $\lambda_1 < 0$ then orbits spiral inwards clockwise - the origin is a **Stable Focus**.

In any number of dimensions we can catalog cases like this, but this very quickly becomes cumbersome and tedious. In general, we don't calculate the JNF, but we do something equivalent, which is calculating the eigenvectors and eigenvalues of A. Then we can draw the phase portrait, and we can then change coordinates easily if we want. So we can explicitly solve the linear parts and understand completely the qualitative behaviors and the relationships of orbits to eigenvectors and eigenvalues of A.

3.3.2 The Non-Linear Part

We want to say that if A is hyperbolic, then the non-linear system is "sort of like" the linear system near the origin, where the h.o.t are small. We get three main types of result:

(I) $\underline{0}$ a sink for the linear part $\Rightarrow$ $\underline{0}$ a sink for the full equation. We can show this with a local Liaponov Function.

(II) Linearisation theorem (Hartman-Grobman): we can map orbits of the linear system onto orbits of the non-linear system in a "nice" way.

(III) Un/Stable manifold theorems: in the non-linear case the un/stable manifold is "nice", and we can learn some techniques to calculate the un/stable manifold.

3.4 Stable and Unstable Manifolds

Example 3.4.1: Consider $\dot{x} = x$, $\dot{y} = -y$. Eigenvalues are ± 1, eigenvectors are the axes. Exactly two orbits tend to $(0, 0)$ as $t \to \infty$ (the y-axis), and exactly two orbits tend to $(0, 0)$ as $t \to -\infty$ (the x-axis). The x and y axes are called **Seperatices**, since they separate $\mathbb{R}^2$ into regions of similar behaviour, made up of orbits of the ODE so the other orbits cannot cross them.

Definition: The **Stable Manifold** $\mathcal{W}^{\mathrm{s}}(x_0)$ of a fixed point x_0 is the set

$$\left\{ x : \lim_{t \to \infty} (\phi(x, t)) = x_0 \right\}$$

Definition: The **Unstable Manifold** $\mathcal{W}^{\mathrm{u}}(x_0)$ of a fixed point x_0 is the set

$$\left\{ x : \lim_{t \to -\infty} (\phi(x, t)) = x_0 \right\}$$

Exercise: Check that the example satisfies the definition.

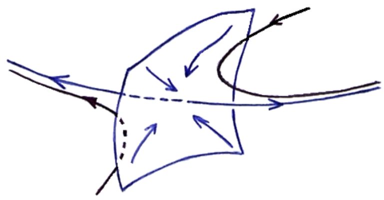

Figure 3.7: The definition holds for the linear and non-linear flows in any number of dimensions. This example is in $\mathbb{R}^3$, where the stable manifold in blue is two dimensional and the stable manifold is one dimensional. Possible flows are in black.

Theorem: For $n > 2$, at a saddle point for hyperbolic fixed points x_0,

$$\dim\left(\mathcal{W}^{\mathrm{s}}(x_0)\right) + \dim\left(\mathcal{W}^{\mathrm{u}}(x_0)\right) = n$$

We often talk about the local manifolds - we are only concerned by orbits close to x_0, denoted by $\mathcal{W}^{\mathrm{s}}_{\mathrm{loc}}(x_0)$ and $\mathcal{W}^{\mathrm{u}}_{\mathrm{loc}}(x_0)$.

Example 3.4.2: Consider $\dot{x} = x$, $\dot{y} = x^2 - y$. We can see that $\{x = 0\}$ is still the stable manifold. We guess that there is an unstable manifold $\mathcal{W}^{\mathrm{u}}(x_0)$ tangent to the x-axis. On the assumption that it exists, try to calculate it. Suppose it has the equation $y(x) = \sum_{n=0}^{\infty} a_n x^n$, i.e. a standard geometric object. We should have $a_0 = a_1 = 0$, since it goes through the origin and is tangent to the x-axis (a_1 is chosen to give the same slope as the appropriate eigenvector). Calculate coefficients using the fact that we have two expressions for $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 1a_1x^2 + 2a_2x + 3a_3x^2 + \ldots \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\dot{y}}{\dot{x}} = \frac{x^2 - y}{x} \\
&= \frac{x^2 - (a_0 + a_1x + a_2x^2 + \ldots)}{x} \\
&= -\frac{a_0}{x} - a_1 + (1 - a_2)x - a_3x^2 - \ldots
\end{aligned}$$

Equating coefficients, we get $a_0 = 0$, $a_1 = -a_1 \to a_1 = 0$ as expected, $2a_2 = 1 - a_2 \Rightarrow a_2 = \frac{1}{3}$, and for $n > 2$, $na_n = -a_n \Rightarrow a_n = 0$. So $y = \frac{1}{3}x^2$ is the equation of the unstable manifold.

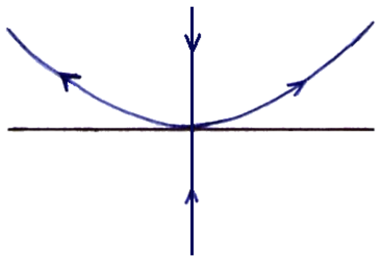

Figure 3.8: The unstable manifold.

Note: Since in this case we can solve the ODE, we can check this result by hand; do this as an exercise. Observe that we can always change coordinates to get a "nice" linear part.

Example 3.4.3: A more general example:

$$\dot{x} = x + xy, \ \dot{y} = x^2 - xy - y$$

This has the same linear part as before, so $\mathcal{W}^{\mathrm{s}}((0,0))$ and $\mathcal{W}^{\mathrm{u}}((0,0))$ are tangent to the y and x axes respectively as before. Consider as before the expressions

$$\mathcal{W}^{\mathrm{u}}\left((0,0)\right) = y(x) = \sum_{n=2}^{\infty} a_n x^n, \ \mathcal{W}^{\mathrm{s}}\left((0,0)\right) = x(y) = \sum_{n=2}^{\infty} b_n y^n$$

and compare the coefficients to this expression:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{x^2 - xy - y}{x + xy}$$

In the case of the unstable manifold we get

$$(2a_2x + 3a_3x^2 + \ldots)(x + xy) = x^2 - y - xy$$
$$(2a_2x + 3a_3x^2 + \ldots)\left(x + x(2a_2x + 3a_3x^2 + \ldots)\right) = x^2 - (1 + x)(2a_2x + 3a_3x^2 + \ldots)$$

after tedious algebraic manipulation (an exercise) we get that

$$y(x) = \frac{1}{3}x^2 - \frac{1}{12}x^3 - \frac{1}{36}x^4 + \ldots + \text{h. o. t.}$$

Do the stable case as an exercise. This is a good approximation of $\mathcal{W}^{\mathrm{u}}((0,0))$ near $(0,0)$, but we don't know in what neighbourhood it is good without doing more work - see Analysis II.

But we need to make sure that the un/stable manifolds actually exist.

3.5 Stable Manifold Theorem (SMT)

We will outline a geometric proof for the 2 dimensional case.

Theorem (Stable Manifold Theorem where $\dim(X) = 2$)**:** Given a fixed (saddle) point x_0 of a $\mathcal{C}^1$ vector field, there are exactly two orbits (in addition to x_0 itself) tending to x_0 as $t \to \infty$. They approach x_0 tangentially to the stable contracting subspace E^{s} of $\mathrm{D}f(x_0)$ one from each side.

$\mathrm{D}f(x_0)$ is the matrix A - the linear part of the vector field. E^{s} is the eigenspace corresponding to the eigenvalues of A with negative real parts; in this 2 dimensional case the eigenvalues are real and $\mathrm{E}^{\mathrm{s}}(x_0)$ is the corresponding eigenvector.

Remark: By reversing time we get the "Unstable Manifold Theorem".

Glendinning's proof is based on a succession of non-linear changes of coordinates that makes the equations "more linear", i.e. the higher order terms become of higher and higher order. Perko's proof uses Picard iteration to get flows converging to the stable manifold. We will do a geometric proof.

Proof. Change coordinates linearly so that x_0 is at the origin and E^{s} and E^{u} are the y and x axes respectively. Draw a box D of size $\pm\varepsilon(> 0)$ around the origin. Label the top T, the bottom B, the left side L and the right side R.

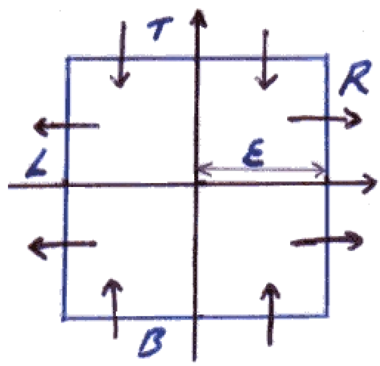

Figure 3.9: The box D.

The equations look like

$$\dot{x} = \lambda x + \text{h. o. t.}$$
$$\dot{y} = -\mu y + \text{h. o. t.}$$

where $\lambda, \mu > 0$. For $\varepsilon > 0$ small enough, the flows enter through T and B and leave through L and R. Define subsets of T as follows:

$$l := \{x \in T : \phi(x,t) \text{ leaves } D \text{ through } L\}$$
$$r := \{x \in T : \phi(x,t) \text{ leaves } D \text{ through } R\}$$

Note that l and r are non-empty (they contains the corners), open (by continuity of the flow) and disjoint (flows can't leave through both L and R). So $l \cup r \neq T$. Thus there is at least one point $z^+ \in T$ such that $\phi(z^+, t)$ does not leave D (can't leave through D or B). D is compact, so $\omega(z^+) \neq \varnothing$. Consider $V(x,y) = \frac{1}{2}(y^2 - x^2)$. $\dot{V} = y\dot{y} - x\dot{x} = -\mu y^2 - \lambda x^2 + \text{h.o.t.} < 0$ for x and y small enough. So within D, V is a Liaponov function. La Salle's invariance principle $\Rightarrow \omega(z^+) = \{0\}$. Check that $V < 0$ for $x^2 > y^2$ does not affect this argument. Similarly $\exists\, z^- \in B$ such that $\omega(z^-) = \{0\}$. We need to argue that z^+ and z^- can't be on the same orbit, since they only go though T or B once and cannot leave D. So there are at least two orbits that tend to the origin as $t \to \infty$. Now we need to show that

(I) Orbits approach tangentially to the y axis and

(II) There are exactly two orbits.

To do this,

(I) Look at the cone C in D, $|x| \leq |y|$.

   (a) Consider
   $$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|x^2|\right) \geq \frac{1}{2}\lambda x^2$$
   for points in $D \setminus C$ the $|x|$ coordinate grows and leaves D through L or R, i.e. $\phi(z^+, t) \subseteq C$.

   (b) Similarly
   $$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|y^2|\right) \leq -\frac{1}{2}\mu y^2 + \text{h.o.t.}$$
   For D small enough and inside C. So the $|y|$ coordinate decreases if you are in C, so $\phi(z^+, t) \subseteq C$ and goes to zero as $t \to \infty$.

   (c) Consider $\gamma(t) = x(t)/y(t)$. Want to show $\gamma(t) \to 0$ as $t \to \infty$ for this orbit. Compute $\dot{\gamma}(t)$:
   $$\dot{\gamma} = \frac{y\dot{x} - x\dot{y}}{y^2} = -(\lambda + \mu)y + g(x, y, \gamma)$$
   where $|g(x,y,\gamma)| \leq h(y) \to 0$ as $y \to 0$. Note that we know already that this orbit stays in C, so γ can't grow exponentially. Hence we can deduce that for this orbit
   $$|\gamma| \leq \frac{|h(y)|}{\lambda + \mu} \text{ as } y \to 0$$
   So $\gamma \to 0$ and the orbit is tangential to the y axis.

(II) Sketch: Suppose there are two orbits staring on T. Then orbits starting in T between them must must remain in between them and thus also tend to 0 as $t \to \infty$. Straight lines joining them should therefore get shorter, but this contradicts the fact that the x direction is increasing, so there must only be one orbit coming in from the top. ∎

Example 3.5.1: Application of the SMT to the damped pendulum: there are exactly two orbits tending to the saddle point. We know by La Salle's invariance principle that all orbits tend to a fixed point, so all other orbits have to tend to the stable fixed point.

How does this work in dimensions greater than two?

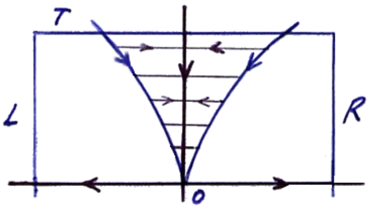

Figure 3.10: Illustrating part (II) of the proof.

Theorem (Stable Manifold Theorem where $\dim(X) > 2$)**:** If $\exists$ hyperbolic fixed point x such that $\mathrm{D}f(x)$ has m eigenvalues with negative real part and $n-m$ eigenvalues with positive real part (with $0 < m < n$), define E^{s} as the linear subspace spanned by eigenvectors of $\mathrm{D}f(x)$ with $\Re(\lambda) < 0$, and E^{u} as the linear subspace spanned by eigenvectors of $\mathrm{D}f(x)$ with $\Im(\lambda) > 0$. Then $\exists\, m$ dimensional stable manifold $\mathcal{W}^{\mathrm{s}}(x)$ tangential to E^{s} on which orbits tend to x as $t \to \infty$, and $\exists n-m$ dimensional unstable manifold $\mathcal{W}^{\mathrm{u}}(x)$ tangential to E^{u} on which orbits tend to x as $t \to -\infty$.

Note: The SMT does not say *how* orbits approach x on $\mathcal{W}^{\mathrm{s}}(x)$, only that they do; for example if $\dim(\mathcal{W}^{\mathrm{s}}(x)) = 2$ and the eigenvalues are $-\lambda \pm i\omega$ then orbits do not necessarily spiral into x. $\mathcal{W}^{\mathrm{s}}(x)$ and $\mathcal{W}^{\mathrm{u}}(x)$ don't have to be orthogonal.

3.6 Hartman-Grobman Theorem

The SMT only concerns orbits going to x. Can we match up orbits of non-linear and linear systems?

Theorem: Suppose x is a hyperbolic fixed point. Then $\exists$ a neighbourhood U of x and a homeomorphism $h : U \to U$ such that $\forall\, x \in U$, there is an interval of time $T(x) \ni 0$ such that $h(\phi(x,t)) = e^{At}h(x)\,\forall\, t \in T(x)$, where $A = \mathrm{D}f(x)$.

h gives a 1-1 map between points y and the orbits through them to points $h(y)$ and the orbits through them. In some senses this a strong theorem; if all orbits tend to x in the linear system, then all orbits in the extended non-linear system do so; a sink in a linear system is a sink in the non-linear system. In other senses it is weak - h is not a diffeomorphism, so we could map star nodes into focii.

Example 3.6.1: See question 4 on example sheet 2. This does not contradict the theorem, but we still can't say anything about the nature of the orbits within U and $h(U)$.

Remark: If $f \in \mathcal{C}^2$, then we can find $h \in \mathcal{C}^1$, but even if $f \in \mathcal{C}^\infty$ then we can't necessarily have $h \in \mathcal{C}^2$.

Corollary: If the linear part $\mathrm{D}f$ has $\Re(\lambda) \le -a < 0$ for some $a > 0$ at a fixed point x_0, then for the non-linear system there is a norm $\|\cdot\|$ such that $\|\phi(x,t) - x_0\| \le e^{-at}\|x - x_0\|$.

Proof. Define $\|\cdot\|$ such that $\|x\|$ is a local Liaponov function. ■

3.7 Non-Hyperbolic Fixed points

If $\dot{x} = Ax + \mathrm{h.o.t.}$ and $\Re(\lambda) = 0$ for some eigenvalues λ of A, when we perturb the system they bifurcate into one or more stable or "unstable" fixed points, so we don't usually worry about them.

Examples 3.7.1:

(A) $\dot{x} = \omega y$, $\dot{y} = -\omega x$, i.e. $\lambda = \pm i\omega$. Adding non-linear terms can make the origin an un/stable focus (see figure 1.4), but worse can also happen; if $\dot{\theta} = \omega$, $\dot{r} = r^2\sin(1/r^2)$, then the circles become alternately stable and unstable periodic orbits accumulating on 0.

(B) If $\dot{x} = x^2$, $\dot{y} = -y$ we get a saddle node, but if we insist that the quadratic terms are zero, we get a non-linear saddle.

(C) Two real eigenvalues being zero - when $A \neq 0$:

(1) $\dot{x} = y$, $\dot{y} = x^2$ gives a cusp.

(2) $\dot{x} = y$, $\dot{y} = 4xy - x^3$ gives something more interesting. Check that $y = x^2/(2 \pm \sqrt{2})$. The upper part is the parabolic domain, and the lower is the elliptic domain.

(D) When $A = 0$ there are two cases:

(1) $\dot{x} = x^3 - 2xy$, $\dot{y} - 2xy^2 - y^3 \Leftrightarrow \dot{r} = r^3 \cos(2\theta)$, $\dot{\theta} = 2r^2 \sin(2\theta)$.

(2) $\dot{x} = xy + x^2$, $\dot{y} = \frac{1}{2}y + xy$

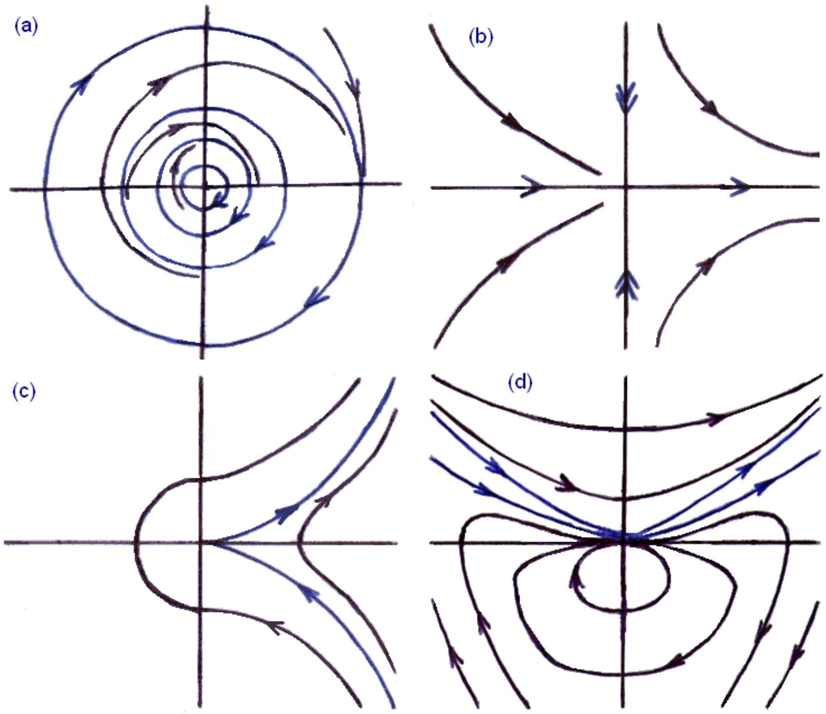

Figure 3.11: Examples 3.7.1(A), (B) and (C): (a) As the periodic orbits (in blue) get closer to the origin they get closer together; similar to the topologist's sine curve. Note the alternating stability of the periodic orbits. (b) The Saddle Node. (c) The cusp. (d) The two lines in blue are the invariants $y = \frac{x^2}{2 \pm \sqrt{2}}$; above them is the hyperbolic domain, in between is the parabolic domain and below is the elliptic domain.

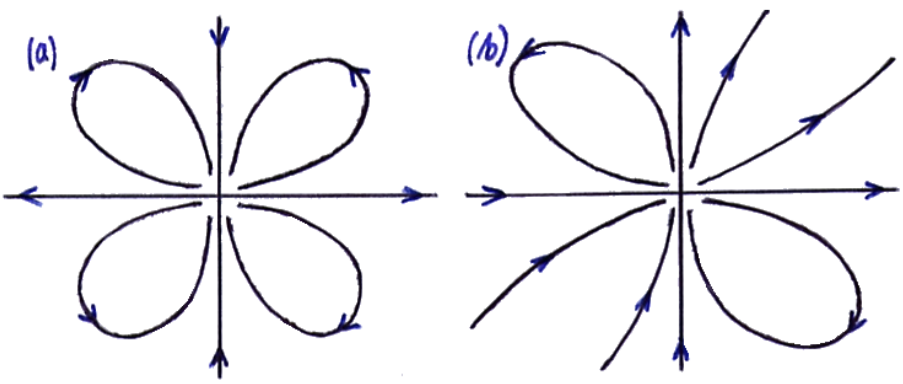

Figure 3.12: Example 3.7.1(D). (a) shows 1. while (b) shows 2.

Remark:

- We can completely classify such as these, but it is complicated.
- Behaviour occurs in sectors separated by seperatices, in which behaviour is either parabolic, hyperbolic or elliptic.
- There are a finite number of sectors: if $\dot{x} = P_m(x, y)$ and $\dot{y} = Q_m(x, y)$, P and Q polynomials where m is the lowest degree term in P and Q, then the number of sectors is $\leq 2(m + 1)$ (maximum number of zeros for $\theta = 0$ as $r \to 0$ in polar coordinates).
- The number of elliptic sectors minus the number of hyperbolic sectors is even. Proof is by the index theorem.

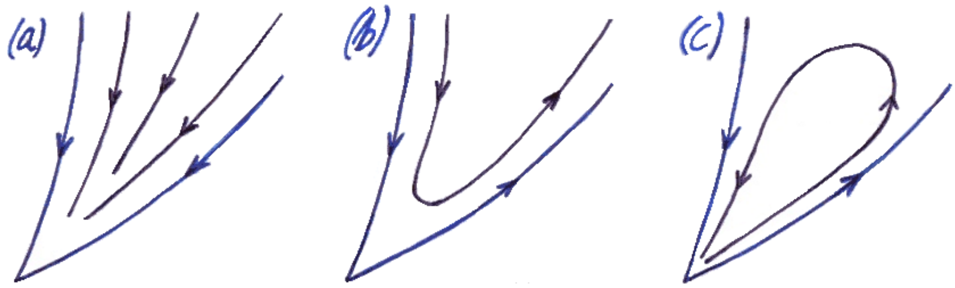

Figure 3.13: (a) behaviour in the parabolic domain. (b) the hyperbolic domain. (c) the elliptic domain.

Chapter 4

Periodic Orbits

We will mostly be looking in $\mathbb{R}^2$, since periodic orbits are closed curves, defining an inside and an outside by the JCT. Other orbits can't cross the periodic orbit, so regions inside and outside are invariant under flows, and they don't change size if they are finite. There are three main theorems of interest:

1. Poincaré-Bendixson Theorem
2. Dulac's Criterion
3. Index Theorem

4.1 The Poincaré-Bendixson Theorem

Theorem: $\dot{x} = f(x)$, $X = \mathbb{R}^2$. If $\phi(x,t) \in \Upsilon \,\forall\, t \geq 0$ with Υ compact, then either

(a) $\omega(x)$ contains a fixed point, or

(b) $\omega(x)$ is a periodic orbit.

Remark: So if there are no fixed points in the region then $\omega(x)$ is a periodic orbit. The theorem is often stated for annuli, but we do a more general proof.

Proof. Since Υ is compact, $\omega(x) \neq \varnothing$. Suppose $y \in \omega(x)$. Consider $\phi(y,t)$ and the set $\omega(y)$, which is non-empty, so pick $z \in \omega(y)$. Either z is a fixed point, in which case $z \in \omega(y) \subseteq \omega(x)$ and we are in (a) of the theorem, or z is not a fixed point and $f(z) \neq 0$. We can draw a small section Σ through the flow containing z such that the flow is transverse to Σ.
We will show that y and z lie on a periodic orbit. Suppose they did not. The orbit through z, $\phi(z)$, must keep returning to Σ, crossing it in the same direction each time. This follows from the fact that $z \in \omega(y)$, so the orbit from y comes arbitrarily close to z at an increasing sequence of times $t_i \to \infty$. Suppose $z \neq z_1$, where z_1 is the first time that $\phi(z) \cap \Sigma \neq \varnothing$. The orbit from z to $z_1 \cup [z, z_1] \subseteq \Sigma$ is a closed curve, so by the JCT there is an inside and an outside.
So either orbits can cross Σ into the inside and then can't escape, or they can start inside and be exiled, never to return. So in both cases the orbit through y eventually comes close enough to z that it eventually gets trapped inside and can't come close to z again or gets expelled and can't come close to z again. But $z \in \omega(y)$, so we must have $z = z_1$.
Either y lies on this orbit (we want to show this) or $\phi(y,t)$ makes a sequence of intersections with Σ; $y_1, y_2, \ldots \to z$. We can show by a similar argument that $\{y_i\}_{i=1}^{\infty}$ is a monotonic sequence of distinct points - a contradiction unless $\forall\, i$, $y_i = z$ and y is on the periodic orbit. This relies on the fact that $y \in \omega(x)$.
We have proved that $\omega(y)$ is a periodic orbit through y and z. We need to show that $\omega(x)$ is this periodic orbit. A similar argument implies that either x is a part of the periodic orbit or $\phi(x,t)$ intersects Σ in a monotonic sequence of points $x_1, x_2, \ldots \to z$ all distinct, so $\omega(x)$ is the periodic orbit. ∎

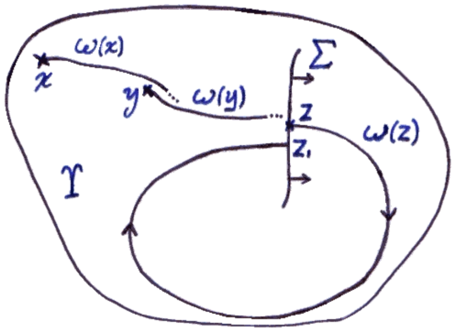

Figure 4.1: Illustration of the proof.

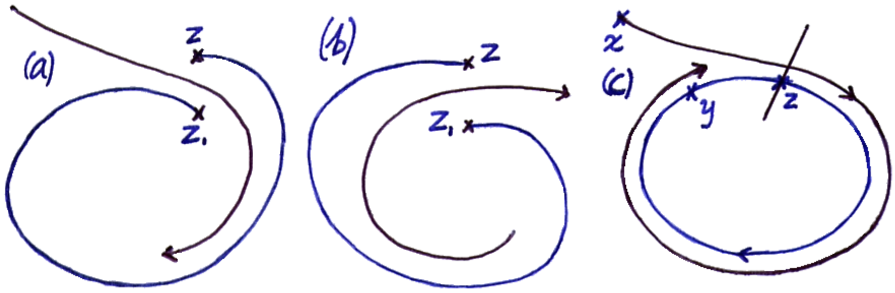

Figure 4.2: (a) orbits can get in but not out, or (b) they get out but can't return. (c) The final part of the proof.

4.2 Dulac's Criterion or the Negative Divergence Test

Recall that $\mathrm{div}(f) := \mathrm{Tr}(\mathrm{D}f) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$.

Theorem: Given an infinitesimal volume element $V = \mathbb{R}^n$, then

$$\frac{\mathrm{d}V}{\mathrm{d}t} = V\,\mathrm{div}(f)$$

Note: This doesn't imply that volume elements can shrink to a point (except at infinity).

Proof. Assume that V is a parallelepiped with edges $\{\underline{b}_1, \dots, \underline{b}_n\}$. Write the matrix $B := (\underline{b}_1 \cdots \underline{b}_n)$. Then $\mathrm{Vol}(V) = \det(B)$. Each $\underline{b}_i$ evolves under the flow: $\dot{\underline{b}}_i = \mathrm{D}f\underline{b}_i + \text{h.o.t.}$.

$$\begin{aligned}
\det(B(t+h)) &= \det\left(B(t) + h\mathrm{D}fB(t) + \text{h.o.t.}\right)\\
&= \det\left(B(t)(I + h\mathrm{D}f + \text{h.o.t.})\right)\\
&= \det(B(t))\det(I + h\mathrm{D}f + \text{h.o.t.})\\
&= V(t)(1 + h\,\mathrm{Tr}(\mathrm{D}f) + h^2\cdots)\\
&= V(t)(1 + \mathrm{div}(f) + \text{h.o.t.})
\end{aligned}$$

So $V(t+h) - V(t) = hV(t)\,\mathrm{div}(f) + \text{h.o.t.}$ so

$$\frac{V(t+h) - V(t)}{h} = V(t)\,\mathrm{div}(f) + \text{h.o.t.} \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t} = V\,\mathrm{div}(f) \qquad \blacksquare$$

Theorem: In $\mathbb{R}^2$, if f is $\mathcal{C}^1$, $\mathrm{div}(f) < 0$ everywhere, then there is no invariant set of non-zero finite area.

Proof. If a set is invariant, then its volume V stays the same, i.e. $\dot{V} = 0$, but this contradicts $\dot{V} = -\mathrm{div}(f)V < 0$ for all infinitesimal volume elements. $\blacksquare$

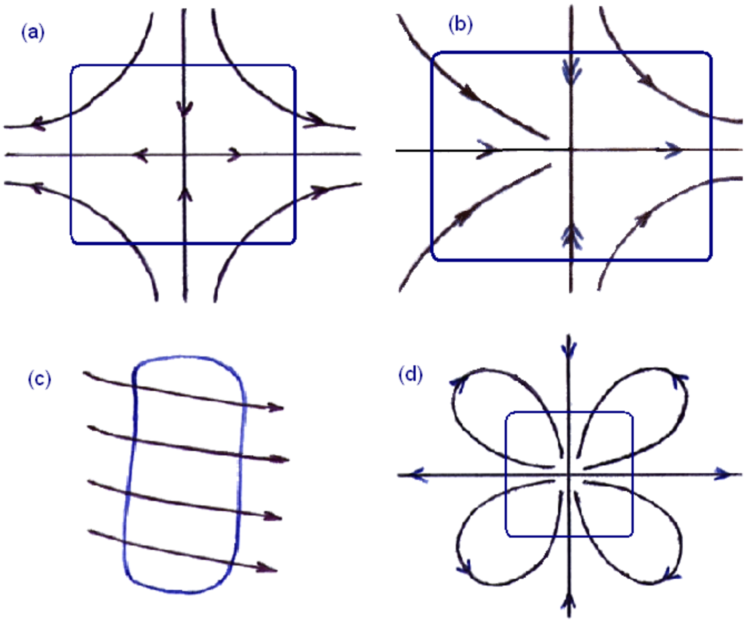

Figure 4.3: (a) The saddle has index -1. (b) The saddle node has index 0. (c) An open region not containing a fixed point has index 0. (d) This non-hyperbolic fixed point has index 3.

Corollary: There cannot be a periodic orbit in $\mathbb{R}^2$ if $\text{div}(f) < 0$, since the area within would be invariant.

Note: We get the same results with $\text{div}(f) > 0$. If $X \neq \mathbb{R}^n$ then be careful: Dulac's Criterion does not rule out a periodic orbit around the girth of $\mathbb{R}^1 \times \mathcal{S}^1$ for example. It does rule out:

(I) Periodic orbits which don't encircle the cylinder.

(II) Two or more periodic orbits which do encircle the cylinder.

The Corollary is true if the phase space X is such that periodic orbits enclose areas of finite volume.

Example 4.2.1: The damped pendulum with constant torque is governed by

$$\ddot{\theta} + k\dot{\theta} + \sin(\theta) = F$$

With $F > 0$. Write $\dot{\theta} = p$, $\dot{p} = F - kp - \sin(\theta)$ and consider $V = \frac{1}{2}p^2\cos(\theta)$. Use $\dot{V} = Fp - kp^2$ to show that trajectories all tend to some bounded region. V is not a Liaponov function, so we can have $\dot{V} > 0$, but $\dot{V} < 0$ if $p \notin \left[0, \frac{F}{k}\right]$, "so orbits tend to $p = 0$ from below and to $p = \frac{F}{k}$ from above". Look for the maximum value of V on the boundary of the region where $V \geq 0$. $\max\{V\} = \frac{1}{2}\left(\frac{F}{k}\right)^2 + 1$. The region where $V = \max\{V\}$ is

$$\frac{1}{2}p^2 - \sin(\theta) < \frac{1}{2}\left(\frac{F}{k}\right)^2 + 1$$

From the work on boundary functions, we may conclude that all orbits eventually enter and remain within this region. What about fixed points? $p = 0$, $\theta = \arcsin(F)$, so $F \in [0, 1) \Rightarrow \exists$ 2 solutions, $F = 1 \Rightarrow \exists!$ solution, and $F > 1 \Rightarrow \nexists$ solutions, so by the PBT there is at least one periodic orbit inside the region $V < V_{\text{max}}$. $\text{div}(f) = 0 - k < 0$, so by Dulac's Criterion, there are no periodic orbits other than the one encircling the cylinder. Check that the periodic orbit goes around the cylinder only once.

Generalisation of Dulac's Criterion: Suppose there is a weighting $\rho : \mathbb{R}^n \to \mathbb{R}^+$, and $\text{div}(\rho f) < 0$. Then by the same argument as before, there is no invariant set of non-zero finite volume in the flow. $\rho = 1$ gives the usual criterion ($\rho f(x) := \rho(x) f(x)$).

Example 4.2.2 (Lotka-Voltera Equations)**:** $\dot{x} = x(A - a_1x + b_2y)$, $\dot{y} = y(B - a_2y + b_2x)$, $a_1, 1_2 > 0$. Check that the positive orthant is invariant. Normally $\text{div}(f) = A - 2a_1x + b_1y + B - 2a_2y + b_2x$, which doesn't have a well defined sign in the positive orthant $\{(x, y) : x, y > 0\}$, so consider $\rho : (x, y) \mapsto \frac{1}{xy}$. Now

$$\begin{aligned}\text{div}(\rho f) &= \text{div}\left(\frac{A - a_1x + b_1y}{y}, \frac{B - a_2y + b_2x}{x}\right) \\ &= -\frac{a_1}{y} - \frac{a_2}{x} < 0 \qquad\qquad (\forall\, x, y > 0)\end{aligned}$$

The Generalised Dulac's Criterion says there are no periodic orbits in the flow.

4.3 Poincaré Index in $\mathbb{R}^2$

Definition: The **Poincaré Index** $\mathrm{I}(\Gamma)$ of a closed curve Γ is the number of times the vector field f rotates anticlockwise as you go once around Γ in an anti-clockwise direction. $\mathrm{I}(\Gamma)$ is always an integer.

Example 4.3.1:

(1) The index of a periodic orbit Γ is $+1$.

(2) A loop Γ around a saddle has index -1; see figure 4.3(a).

(3) A loop around a saddle node has index 0 (figure 4.3(b)).

(4) A loop around an open space has index 0 (figure 4.3(c)).

(5) A loop around the fixed point in figure 4.3(d) has index 3.

Exercise: Check that $\mathrm{I}(\Gamma_1 \cup \Gamma_2) = \mathrm{I}(\Gamma_1) + \mathrm{I}(\Gamma_2)$, and that if Γ is deformed the index only changes if it moves through a fixed point.

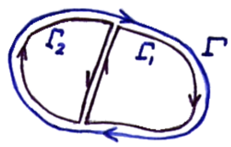

Figure 4.4: $\mathrm{I}(\Gamma_1 \cup \Gamma_2) = \mathrm{I}(\Gamma_1) + \mathrm{I}(\Gamma_2)$.

Theorem: The index of a closed curve Γ is the sum of the indices of the fixed points inside Γ

Proof. Sketch. Assume fixed points are isolated to simplify the argument. Consider a closed curve Γ_i around a fixed point x_i. Deform the Γ_i like this:

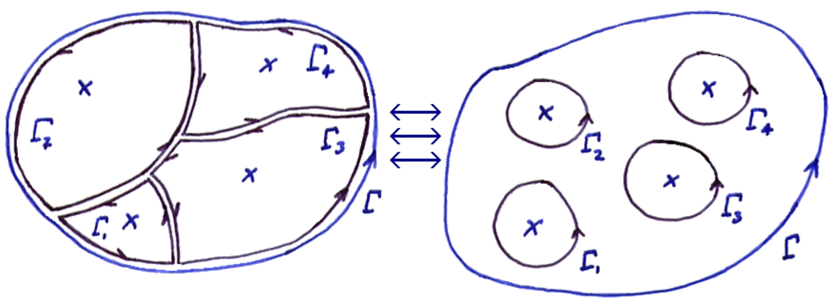

Figure 4.5: The two diagrams are equivalent.

Use the exercise: each Γ_i has the same index as when it was smaller, and $\mathrm{I}(\Gamma) = \sum_i \mathrm{I}(\Gamma_i)$ so $\mathrm{I}(\Gamma) = \sum_i \mathrm{I}(x_i)$. We reach the following conclusions:

(I) A periodic orbit with index $+1$ must enclose at least one fixed point (in $\mathbb{R}^2$).

(II) If a fixed point in $\mathbb{R}^2$ is hyperbolic, the index is ± 1 (check).

(III) For hon-hyperbolic fixed points the index gives a clue as to what will happen if we perturb the flow. ■

This gives an alternative proof for the pendulum: $F > 1 \Rightarrow \nexists$ periodic orbits not encircling the cylinder since $\nexists$ fixed pints.

4.4 Stability of Periodic Orbits

This is usually very hard to calculate, unless we have explicit solutions. One can use standard software on examples to compute approximations to the quantities concerned. The main idea has two approaches. Do orbits get closer to the periodic orbit?

(I) Think of the distance d from the periodic orbit and derive an equation $\dot{d} = \lambda d + \text{h. o. t.}$, so $\lambda > 0 \Rightarrow$ the orbit goes away from the periodic orbit and $\lambda < 0 \Rightarrow$ contracting towards the periodic orbit. $\lambda = 0$ gives the non-hyperbolic case. Such λ are the Liaponov Exponents, but they are "pointwise", so we need to find an average over the whole orbit. λ is measured "per unit time".

(II) Draw a section Σ through the periodic orbit with initial distance d_0 from the orbit to the periodic orbit. Orbits return at distance d_1, with $d_1 = \mu d_0$. $\mu < 0 \Rightarrow$ they get closer, $\mu > 1 \Rightarrow$ they get further away, $\mu = 1 \Rightarrow$ the non-hyperbolic case. μ are the Floquet Multipliers, measured "per revolution".

The two are related: $\lambda = \frac{1}{T}\log(\mu)$ and $\mu = e^{\lambda T}$ (where T is the period of the orbit). More carefully, in $\mathbb{R}^{n\geq 2}$, choose a small section Σ of dimension $n-1$ transverse[1] to the orbit. Σ is called the "Poincaré Section". The periodic orbit is Γ, and define $\gamma_0 := \Sigma \cap \Gamma$.

Definition: Define the **First Return Map** $\Phi : \Sigma \to \Sigma$ by $\Phi(x) := \{\phi(x,T) : T > 0$ is the smallest time such that $\phi(x,T) \in \Sigma\}$. So γ_0 is a fixed point for Φ, i.e. $\Phi(\gamma_0) = \gamma_0$.

There may be other points x near γ_0 such that $\Phi(x) = x$, in which case these are also on periodic orbits but with different periods. The discussion of the stability of the periodic orbit is in terms of the stability of γ_0 as a fixed point of Φ. Consider $\delta \in \Sigma$:

$$\Phi(\gamma_0 + \delta) = \gamma_0 + \delta \mathrm{D}\Phi + \text{h. o. t.}$$

Definition: The **Floquet Multipliers** $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $\mathrm{D}\Phi$.

Lemma: The above definition is independent of the choice of Σ.

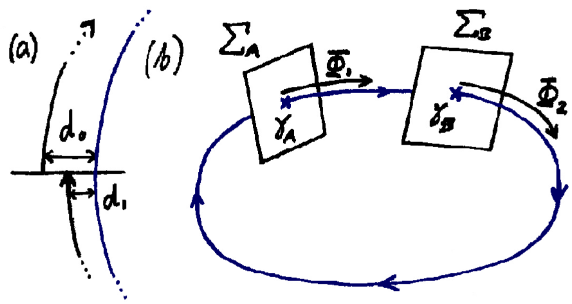

Figure 4.6: (a) The idea of Floquet Multipliers. (b) Illustration of the lemma.

[1] Usually orthogonal, but not tangential to γ_0 will do.

Proof. suppose we have two sections Σ_A and Σ_B. Define $\Phi_1 : \Sigma_A \to \Sigma_B$ and $\Phi_2 : \Sigma_B \to \Sigma_A$. Then $\Phi_A : \Sigma_A \to \Sigma_A = \Phi_2 \circ \Phi_1$, and $\Phi_B : \Sigma_B \to \Sigma_B = \Phi_2 \circ \Phi_2$. Moreover, $\mathrm{D}\Phi_A = \mathrm{D}(\Phi_2 \circ \Phi_1)$ and $\mathrm{D}\Phi_B = \mathrm{D}(\Phi_1 \circ \Phi_2)$ have the same eigenvalues, so the μ_i are well-defined. ■

Definition: A periodic orbit $\gamma(t)$ is **Hyperbolic** if all of the $n-1$ Floquet Multipliers satisfy $|\mu_i| \neq 1$. If $\gamma(t)$ has all $|\mu_i| < 1$ it is a **Sink**. If at least one $|\mu_i| > 1$ then $\gamma(t)$ is **Unstable**, since in at least one direction the orbit moves away from the periodic orbit.

Theorem: If $\gamma(t)$ is a sink then it is asymptotically stable.

Note: we need $|\cdot|$ since in $\mathbb{R}^{n\geq 3}$ we can have $\mu < 0$, i.e. the orbit can go "in" and "out" of the loop.

Definition: for a periodic orbit $\gamma(t)$ with Floquet Multipliers $\mu_1, \ldots, \mu_{n-1}$, define **Liaponov Exponents** $\lambda_1, \ldots, \lambda_{n-1}$ by $\lambda_i := \frac{1}{T}\log(|\mu_i|)$, where T is the period of the periodic orbit.

Theorem: In $\mathbb{R}^2$, μ is "the area between d_1 and the periodic orbit over the area between d_0 and the periodic orbit", i.e.

$$\mu = \exp\left(\int_0^T \mathrm{div}\Big(f\left(\gamma(t)\right)\Big)\mathrm{d}t\right)$$

Example 4.4.1: In the damped pendulum, $\mathrm{div}(f) = -k$ everywhere, so for a periodic orbit of period T, $\mu = e^{-kT} < 1$ so the periodic orbit is stable in terms of Floquet multipliers.

Note: For $\mathbb{R}^{n>2}$ we have instead

$$\prod_{i=1}^{n-1} |\mu_i| = \exp\left(\int_0^T \mathrm{div}\Big(f\left(\gamma(t)\right)\Big)\mathrm{d}t\right)$$

4.5 Van de Pol Oscillators

These were the first chaotic oscillators to appear in the literature. We do not study the chaotic case, which occurs when one adds oscillating forcing. We may learn some additional techniques which are useful in general, and we will introduce the idea of varying parameters. The equation is

$$\ddot{x} + (x^2 - \beta)\dot{x} + x = 0$$

with $\beta > 0$, which we will vary. Note that if the $\dot{x}$ term is missing, i.e. $(x^2-\beta)\dot{x} = 0$, then we have simple harmonic motion. If $x^2 > \beta$ we have damping, and if $x^2 < \beta$ there is negative damping or forcing. So is there an oscillation near $x^2 \approx \beta$? Sort of; we will show this. We will consider various forms of the equations, which will be helpful for $\beta \ll 1$ and $\beta \gg 1$. We want to get the equations into the **Lienard Form**:

$$\dot{x} = y - R(x) \qquad \dot{y} = -x$$

Note that the Van de Pol equation is a special case of equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

So write $y = \dot{x} + F(x)$ where $F(x) = \int_0^x f(s)\mathrm{d}s$. Then $\dot{y} = \ddot{x} + \dot{x}f(x) = -g(x)$ and $\dot{x} = y - F(x)$. So we can get the "general" Van de Pol equation in Lienard form:

$$\begin{aligned}\dot{x} &= y - F(x)\\ \dot{y} &= -g(x)\end{aligned}$$

The particular Van de Pol equation in Lienard form we're looking at is

$$\begin{aligned}\dot{x} &= y + \beta x - \frac{x^3}{3}\\ \dot{y} &= -x\end{aligned}$$

It turns out to be convenient to scale x so that the oscillations expected for $x^2 \approx \beta$ occur for $X^2 \approx 1$. Substitute $X\sqrt{\beta} = x$ to get

$$\ddot{X} + \beta(x^2 - 1)\dot{X} + X = 0$$

Now replace X with x to get $f(x) = \beta(x^2 - 1)$, $F(x) = \beta\left(\frac{x^3}{3} - x\right)$, $g(x) = x$ so in Lienard form we have

$$\begin{aligned}\dot{x} &= y - \beta\left(\frac{x^3}{3} - x\right)\\ \dot{y} &= -x\end{aligned}$$

For $\beta \ll 1$ we will learn the method of averaging to show the existence of a stable periodic orbit. When $\beta \gg 1$ there are stable relaxation oscillators, which are rectangular and "non-sinusoidal" periodic orbits. For equations of this type we argue that $\exists!$ stable periodic orbit $\forall\, \beta > 0$. Consider the function $H : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$, $H : (x, y) \mapsto \frac{1}{2}(x^2 + y^2)$. Then

$$\dot{H} = x\dot{x} + y\dot{y} = xy - \beta\left(\frac{x^4}{3} - x^2\right) - xy = -\beta\left(\frac{x^4}{3} - x^2\right)$$

Note: In the general case,

$$H = \frac{y^2}{2} + \int_0^x g(s)\mathrm{d}s,\ \dot{H} = y\dot{y} + g(x)\dot{x} = -yg(x) + g(x)(y - F(x)) = -g(x)F(x)$$

We now look at the two cases:

(I) $\beta \ll 1$. If $\beta = 0$ we get simple harmonic motion $\dot{x} = y$, $\dot{y} = -x$ with period 2π. If $\beta \neq 0$, solutions will move close to solutions of the $\beta = 0$ case, at least for finite time periods.

$$H = \frac{1}{2}(x^2 + y^2),\ \dot{H} = -\beta\left(\frac{x^4}{3} - x^2\right)$$

So $\dot{H}$ is small if $\beta \ll 1$. For convenience, consider the staring position $(x, 0)$ on the x-axis by using its H value: $\frac{1}{2}x_0^2 = H_0 \Rightarrow x_0 = \sqrt{2H_0}$. The change in H during 1 revolution will be small:

$$\Delta H(t) = \int_0^t \dot{H}\mathrm{d}t = \int_0^t -\beta\left(\frac{x^4}{3} - x^2\right)\mathrm{d}t$$

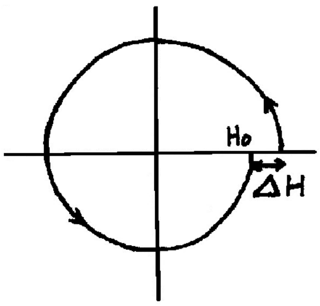

Figure 4.7: Illustrating where the change in H, ΔH, comes from.

We don't know what $x(t)$ is, so approximate for one revolution by solutions of the $\beta = 0$ case - this is the **Method of Averaging**. Take $\int_0^{2\pi}$ (approximate time, can show first order in β). Take $x(t) = x_0\cos(t) = \sqrt{2H_0}\cos(t)$ (the solution in the $\beta = 0$ case). So

$$\begin{aligned}\Delta H(2\pi) &\approx \int_0^{2\pi} -\beta\left(\frac{4H_0^2}{3}\cos^4(t) - 2H_0\cos^2(t)\right)\mathrm{d}t\\ &\approx 2\pi\beta\left(H_0 - \frac{H_0^2}{2}\right) + O(\beta^2)\end{aligned}$$

So $\Delta H = 0$ when $H_0 = 0 \Leftrightarrow x_0 = 0$ or when $H_0 = 2 \Leftrightarrow x_0 = 2$. So we can expect the system to have a periodic orbit passing through $(2, 0)$ on the x-axis with period approximately 2π, correct to the first order in β. Note that $\Delta H < 0$ if $H_0 > 2$ and $\Delta H > 0$ if $H_0 < 2$. So expect the orbit to be stable. In order to check the stability we must calculate the Floquet multiplier, which in turn requires us to compute from the first return map, which is

$$\Phi(H) = H + \Delta H = H + 2\pi\beta\left(H - \frac{H^2}{2}\right) + \mathrm{h.o.t.}$$

So $\Phi'(H) = 1 + 2\pi\beta(1 - H)$. If $H = 2$ then $\Phi'(H) = 1 - 2\pi\beta$ so $\mu = 1 - 2\pi\beta \Rightarrow \mu = 1$ if $\beta = 0$ (which is what we would expect; the same periodic orbit for simple harmonic motion; non-hyperbolic and neutrally stable). $\beta > 0 \Rightarrow \mu < 1$ so the periodic orbit is stable. The Liaponov exponent is

$$\frac{1}{2\pi}\log(1 - 2\pi\beta) \approx -\beta + O(\beta^2)$$

Note: General method: given a small parameter β and known solutions for $\beta = 0$, follow the above method.

(II) $\beta \gg 1$. We already expect interesting behaviour near $x \approx 1$. It is convenient to change the y coordinate so that we have $y \approx 1$. The current equation is $y \approx \beta$ for $\dot{x}$ to change sign. Let $Y = \frac{y}{\beta}$.

$$\dot{Y} = \frac{\dot{y}}{\beta} = -\frac{x}{\beta}, \; \dot{x} = \beta Y - \beta\left(\frac{x^3}{3} - x\right)$$

The new form of the equations is

$$\dot{x} = \beta\left(y - \frac{x^3}{3} + x\right)$$
$$\dot{y} = -\frac{x}{\beta}$$

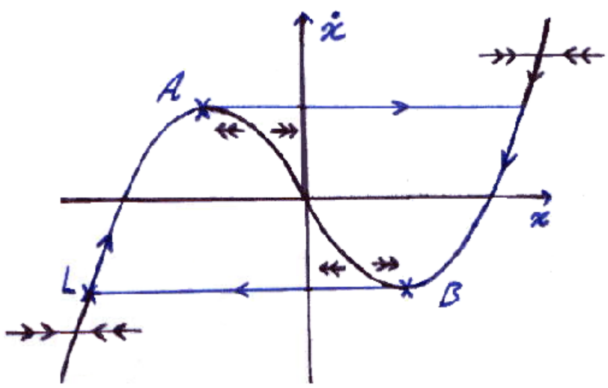

Figure 4.8: The behaviour of the equations. The periodic orbit is in blue.

Look at the curve $y = \frac{x^3}{3} - x$. Away from this, $\dot{x}$ is very large and $\dot{y}$ is very small. Once in this neighbourhood of the curve, drift slowly along ($\dot{y} = -\frac{x}{\beta}$) until you get to points A or B. The curve is “unstable” between A and B. By a loose argument that we expect to see a periodic orbit. Plot x against t to get relaxation oscillations.

Notice that:

(1) We haven’t proved the existence of a stable periodic orbit but will do.

(2) The “loose argument” works since numerical solutions look exactly like this.

(3) Will show, based on loose ideas, how to calculate the period of the orbit.

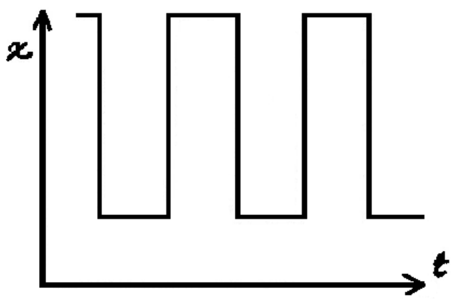

Figure 4.9: Relaxation Oscillations. The flow is moving so fast on the horizontal sections that this sort of "square sine wave" is produced.

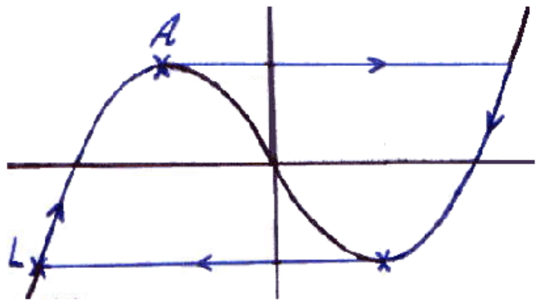

Figure 4.10: $L = (-2, -\frac{2}{3})$, $A = (-1, \frac{2}{3})$. The periodic orbit is in blue.

The period of the orbit is approximately 2 times the slow phases (where the orbit moves near $y = \frac{x^3}{3} - x$, eg from L to A), so

$$T \approx 2\int_L^A \mathrm{d}t = 2\int_{-\frac{2}{3}}^{\frac{2}{3}} \frac{\mathrm{d}y}{\dot{y}} = 2\int_{-\frac{2}{3}}^{\frac{2}{3}} -\frac{\beta}{x}\mathrm{d}y$$

But along this bit of orbit $y = \frac{x^3}{3} - x$ so $\frac{\mathrm{d}y}{\mathrm{d}x}$ along this orbit is $x^2 - 1$. So

$$T \approx 2\int_{-2}^{-1} -\frac{\beta}{x}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = 2\int_{-2}^{-1} -\frac{\beta}{x}(x^2-1)\mathrm{d}x = \beta(3 - 2\log(2))$$

So T varies linearly with β as one would expect. For large β this is a good approximation (but not for small β). For the general $\beta > 0$ we will show that $\exists!$ stable periodic orbit. Consider the general case of

$$\dot{x} = y - F(x)$$
$$\dot{y} = -g(x)$$

where $g : \mathbb{R} \to \mathbb{R}$ such that $g(0) = 0$, $g'(x) > 0 \,\forall\, x \in \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ such that $F(-a) = F(0) = F(b) = 0$, $-a < 0 < b$, $F(x) < 0$ for $x \in (-\infty, -a) \cup (0, b)$, $F(x) > 0$ for $x \in (-a, 0) \cup (b, \infty)$, $F(x) \nearrow \infty$ as $x \nearrow \infty$ and finally $F(x) \searrow -\infty$ as $x \searrow -\infty$, i.e. F looks like a standard cubic. Take $H = \frac{y^2}{2} + \int_0^x g(s)\mathrm{d}s$ so $\dot{H} = -g(x)F(x)$, so $\dot{H} > 0$ in $x \in (-a, b)$ and $\dot{H} < 0$ in $x \in (-a, b)^C$. Draw the nullclines $x = 0$, $\dot{y} = 0$, then divide $\mathbb{R}^2$ into 4 regions:
Step (1): Show that orbits rotate through the four regions.
Step (2): Use $\dot{H} > 0$ for small orbits, and on average $\dot{H} < 0$ for large orbits to conclude that $\exists!$ stable periodic orbit.

Step (1) The only fixed point is at $(0, 0)$. If you start in region 1 you cross into region 2, since x increasing, y decreasing and can only fail to cross if both x and y tend to the same limit $\Rightarrow$ the vector field is zero at this point - a contradiction. Start in region 2 with x and y

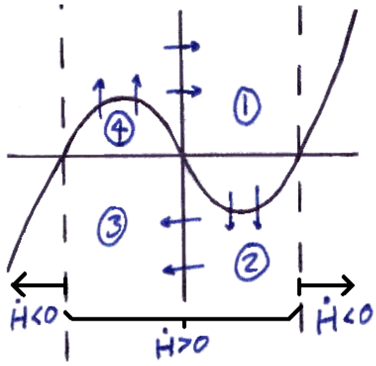

Figure 4.11: The four regions with nullclines for $\dot{x} = 0$ and $\dot{y} = 0$.

decreasing. Again, orbit can't tend to a limit in region 2. Check orbits cannot tend to $-\infty$ in region 2: look at $\frac{\mathrm{d}y}{\mathrm{d}x}$ along orbits. It is $\frac{\dot{y}}{\dot{x}} = -\frac{g(x)}{y-F(x)}$ which gets small as $y \searrow -\infty$, but this contradicts the requirement that $\frac{\mathrm{d}y}{\mathrm{d}x} \nearrow \infty$, so orbits cross into region 3. Similarly, orbits starting in region 3 go into region 4 and orbits starting in region 4 go into region 1.

Step (2) Consider $x \geq 0$. Pick y such that the orbit started at $(0, y_1)$ passes through $(b, 0)$ and crosses the y-axis again at $(0, y_2)$. This lies entirely in the region where $\dot{H} > 0$, so H increases as you go from $(0, y_1)$ to $(0, y_2)$. The same is true for $y > y_1$. Orbits consist of 3 pieces: $z_1 \to B$ ($\dot{h} > 0$), $B \to B'$ ($\dot{H} < 0$) and $B' \to z_2$ ($\dot{H} > 0$). Calculate ΔH along each region (will depend on z_1).

$$
\begin{aligned}
\Delta H_{z_1 \to B} &= \int_0^t \dot{H} \mathrm{d}t = -\int_0^t g(x)F(x)\mathrm{d}t \\
&= -\int_0^b \frac{g(x)F(x)}{y - F(x)}\mathrm{d}x = \int_0^b \frac{g(x)}{1 - \left(\frac{y}{F(x)}\right)}\mathrm{d}x > 0
\end{aligned}
$$

This is positive and decreasing with y. So the larger we take $y = z_1$, the smaller ΔH_1 is (but always > 0). The same argument works for $\Delta H_3 := \delta H_{B' \to z_2}$. For $B \to B'$,

$$
\Delta H_2 = \int \dot{H} \mathrm{d}t = -\int g(x)F(x)\mathrm{d}t = \int_B^{B'} F(x)\mathrm{d}y < 0
$$

from the sign of $F(x > b)$. This integral $\searrow -\infty$ as $B \nearrow \infty$ provided the orbit goes further into the region $x > b$, the larger B is; check that this is true as an exercise. So

$$
\Delta H_{z_1 \to z_2} = \underbrace{\Delta H_1 + \delta H_3}_{>0, \searrow 0 \text{ as } z_1 \nearrow \infty} + \underbrace{\Delta H_2}_{<0, \searrow -\infty \text{ as } z_1 \nearrow \infty}
$$

Look at $\Delta H = \Delta H_1 + \Delta H_2 + \Delta H_3$ versus x (or y_2?). For $b < x$, ΔH is monotonically decreasing, so $\exists!$ point where $\Delta H = 0$.
If F and g are symmetric as in the Van de Pol equation, then we get that $\Delta H = 0$ when $y_2' = -y$ and $\Delta H = 0$ on both sides $\Rightarrow \exists$ periodic orbit in the flow. If F and g are not symmetric then you get two functions ΔH_+ and ΔH_-, both of which have a unique zero, as in figure 4.13(b). We can sometimes compose two functions in such a way that the composition has more than one zero, so in these cases we get more than one periodic orbit. In the case of one orbit $\Delta H > 0$ for $y < y^*$ and $\Delta H < 0$ for $y > y^* \Rightarrow$ the orbit is stable.

Remark:

(1) One can generalise this proof to relax some of the conditions on F and g.

(2) We have mainly looked hard at this example because we can do so. There are many problems for simple vector fields in $\mathbb{R}^2$ where it is very difficult to show the existence of periodic orbits or not, e.g.

$$
\dot{x} = P(x, y), \ \dot{y} = Q(x, y)
$$

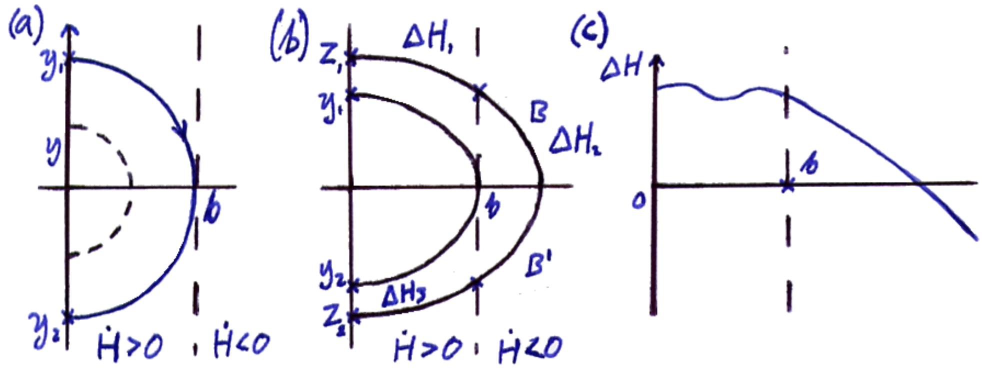

Figure 4.12: (a) When $y < y_1$. (b) When $y > y_1$. (c) Before $(b, 0)$, ΔH is strictly positive, and afterwards ΔH decreases monotonically, so there must be a unique root.

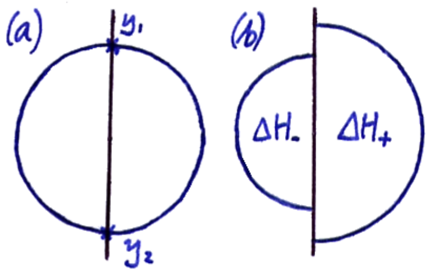

Figure 4.13: (a) The symmetric case. (b) The asymmetric case.

where P and Q are quadratic at worst. How many periodic orbits are there in the flow? This is Hilbert's 16^{th} problem. The answer seems to be 4.

Chapter 5

Bifurcations

Definition: Two flows f and g are **Topologically Equivalent** if the phase portraits are geometrically the same.

Definition: A flow f is **Structurally Stable** if for all flows g in a neighbourhood of f, f and g are topologically equivalent.

What does the space of flows look like?

(1) Are structurally stable flows dense in the space of flows? Yes under certain conditions but in general no! It has been known since the 70's that there are flows where open sets of nearby flows, none of which are structurally stable.

(2) What ways are there for a phase portrait to change as you perturb a flow f? What kind of bifurcations occur?

Typically there are rather a small number of ways to change behaviour - locally and globally:

Locally: near non-hyperbolic fixed points, periodic orbits and similar in higher dimensional spaces.

Globally: connected with occurrence of homoclinic orbits et cetera. There is not much else.

Example 5.0.1: A local bifurcation: The change is near to a fixed point. Eigenvectors at the

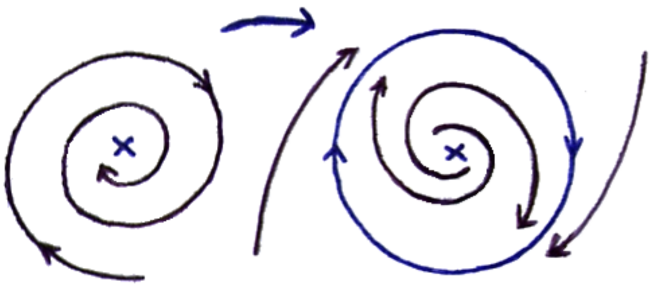

Figure 5.1: A stable node bifurcates to an unstable node with a stable periodic orbit.

fixed point are $\lambda \pm i\omega$ and the bifurcation occurs as λ goes through zero. This is non-hyperbolic at "boundary" between the two cases $0 \pm i\omega$. This is a Hopf Bifurcation, which happens in lots of different contexts, such as PDEs, difference equations etc.

Example 5.0.2: A global bifurcation. See the diagram on the next page.

5.1 Bifurcations in $\mathbb{R}^1$

Consider families $\{f(x, \mu)\}$ of vector fields with $x \in X$ and $\mu \in \mathbb{R}^m$ (usually) a set of variable parameters. We look at lots of "neat" examples, but we need to know how typical these nice

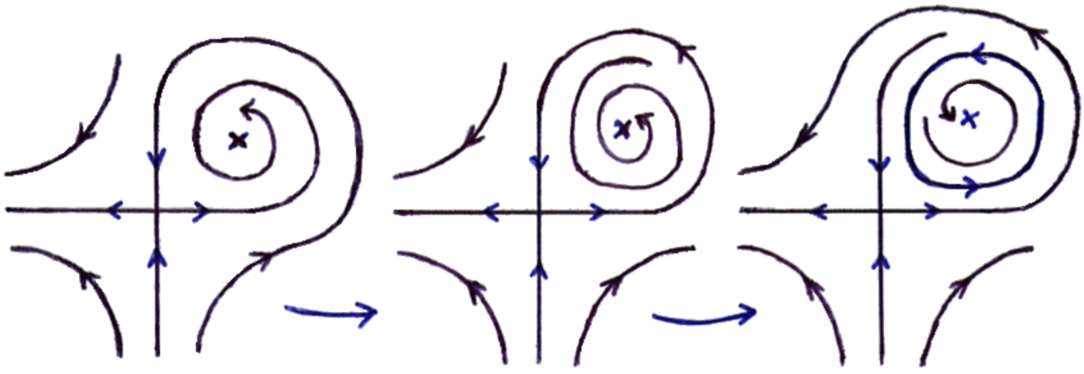

Figure 5.2: Example 5.0.2: A saddle with a stable node → a saddle with a stable node and a homoclinic orbit → a saddle with a stable node and an unstable periodic orbit. Note that the stability of stable points does not change.

examples are.

"**Definition:**" We say that a bifurcation is of **Co-dimension** n you typically see it when varying n parameters.

Examples 5.1.1: $\dot{x} = f(x, \mu)$; $x, \mu \in \mathbb{R}$.

(I) $\dot{x} = \mu - x^2$ gives a **Saddle Node Bifurcation**. There is a non-hyperbolic fixed point at $x = 0$ when $\mu = 0$, no fixed points for $\mu < 0$, a stable fixed point at $x = \sqrt{\mu}$ for $\mu > 0$ and an unstable fixed point at $x = -\sqrt{\mu}$ for $\mu > 0$.

(II) $\dot{x} = \mu x - x^2$ has a non-hyperbolic fixed point at $x = 0$ when $\mu = 0$; stable at $x = 0$ for $\mu < 0$ and $x = \mu$ for $\mu > 0$, unstable at $x = \mu$ for $\mu < 0$ and $x = 0$ for $\mu > 0$. This is called a **Transcritical bifurcation**.

(III) $\dot{x} = \mu x - x^3$ gives the **Pitchfork Bifurcation**, having stable fixed points $x = 0$ for $\mu \leq 0$, $x = \pm\sqrt{\mu}$ for $\mu > 0$ and an unstable fixed point $x = 0$ for $\mu > 0$.

The higher the order of the polynomial, the more complicated the bifurcations get, but also the less likely; we will see that (I) is the simplest and occurs the most often.

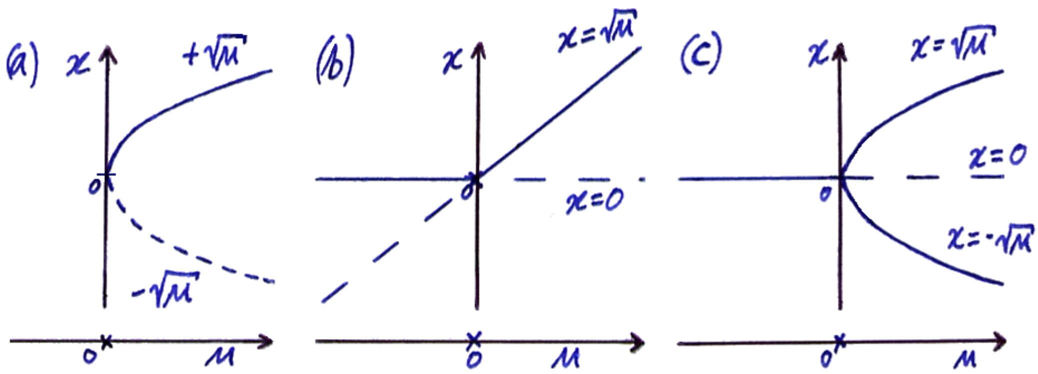

Figure 5.3: Example 5.1.1: (I) a saddle node bifurcation. Note the lack of fixed points for $\mu < 0$. (II) The transcritical bifurcation. (III) The Pitchfork bifurcation.

Suppose we consider a 2-parameter family of flows $\dot{x} = \mu_1 + \mu_2 x - x^2$. If $\mu_1 = \mu_2 = 0$ then there is a non-hyperbolic fixed point at $x = 0$. We get a saddle node bifurcation if $\mu_2 = 0$ and vary μ_1 and a transcritical bifurcation if $\mu_1 = 0$ and μ_2 is varied. Write $y = x - \alpha$. Then

$$\dot{y} = \mu_1 + \mu_2(y + \alpha) - (y + \alpha)^2$$

Set $\alpha = \frac{\mu_2}{2}$ to get

$$\dot{y} = \left(\mu_1 + \frac{1}{4}\mu_2^2\right) - y^2$$

This looks like a member of the family $\mu - x^2$, so if we vary μ_1 and μ_2 in some general way, then $\mu_1 + \frac{1}{4}\mu_2^2$ will sometimes be zero, and will be typically passing through zero to give a saddle node bifurcation. If not, we need to vary μ_1 and μ_2 in such a way that $\mu_1 + \frac{1}{4}\mu_2^2$ reaches zero but doesn't change sign, e.g. $\mu_1 = 0$, μ_2 changes sign as in the transcritical case.

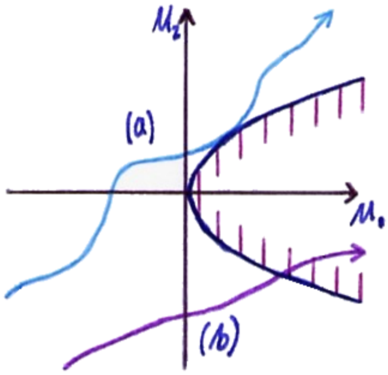

Figure 5.4: There are no fixed points within the red region (only partially shaded for clarity), one on the black line (which is non-hyperbolic), and two in the white region. (a) The cyan blue variation of the parameters μ_1 and μ_2 touches $\mu_2 = \pm\frac{1}{2}\sqrt{\mu_1}$ (the black line) tangentially at one point, at which there is a transcritical bifurcation. Clearly this is less likely to happen than (b), where the purple variation crosses $\mu_2 = \pm\frac{1}{2}\sqrt{\mu_1}$ completely, at which point there is a saddle node bifurcation.

Alternatively, look at the graph above. As we vary μ_1 and μ_2, crossing $\mu_2 = \pm\frac{1}{2}\sqrt{\mu_1}$ gives a saddle node bifurcation, but the only way to get a transcritical bifurcation is if the $\mu_1 - \mu_2$ curve of variation is tangential to $\mu_2 = \pm\frac{1}{2}\sqrt{\mu_1}$, which is much less likely.

Consider $\dot{x} = \mu_1 + \mu_2 x + \mu_3 x^2$. There is a non-hyperbolic fixed point at zero when $\underline{\mu} = \underline{0}$. Scale time $T = \gamma t$, so

$$\frac{\mathrm{d}x}{\mathrm{d}T} = \frac{\mu_1}{\gamma} + \frac{\mu_2}{\gamma}x + \frac{\mu_3}{\gamma}x^2$$

set $\gamma = -\mu_3$ to get

$$\frac{\mathrm{d}x}{\mathrm{d}T} = -\frac{\mu_1}{\mu_3} - \frac{\mu_2}{\mu_3}x - x^2$$

so this is now in the previous form with μ_1 replaced by $-\frac{\mu_1}{\mu_3}$ and μ_2 replaced by $-\frac{\mu_2}{\mu_3}$. So having a different coefficient in front of x^2 changes very little, provided it is non-zero. The general case is $\dot{x} = f(x, \mu)$, $\mu \in \mathbb{R}$. $\mu = 0 \Rightarrow \exists$ a non-hyperbolic fixed point at $x = 0$, so $f(0,0) = 0$ and $\mathrm{D}f(0,0) = 0$. Then for small x and μ, take the Taylor expansion of f:

$$\begin{aligned}
f(x,u) &= f(0,0) + x\mathrm{D}f(0,0) + \mu\frac{\partial f}{\partial \mu}(0,0) + \frac{x^2}{2}\mathrm{D}f(0,0) + x\mu\frac{\partial^2 f}{\partial x \partial \mu}(0,0) + \frac{\mu^2}{2}\frac{\partial^2 f}{\partial \mu^2}(0,0) + \text{h.o.t.} \\
&= \left[\mu\frac{\partial f}{\partial \mu}(0,0) + O(\mu^2)\right] + x\left[\mu\frac{\partial^2 f}{\partial x^2}(0,0)\right] + x^2\left[\frac{1}{2}\mathrm{D}^2 f(0,0)\right] + \text{h.o.t.} \\
&= \mu_1 + \mu_2 x + \mu_3 x^2 + \text{h.o.t.}
\end{aligned}$$

so we understand this provided the various co-ordinate transforms are allowed, i.e. for $\gamma \neq 0$ we need $\mathrm{D}^2 f(0,0) \neq 0$; this is not satisfied by the pitchfork for example.

Remark: Sometimes a system has a symmetry that you want to take as fixed. This forces derivatives such as $\partial^2 f/\partial x \partial \mu$ to be zero, and we can get special bifurcations. Otherwise, if you look at one parameter families you typically see saddle node bifurcations.

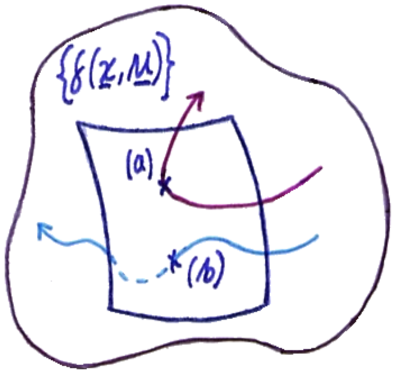

Figure 5.5: The “manifold” is the set of non-hyperbolic fixed points. In (a) the red line has a transcritical bifurcation where it touches the manifold tangentially and in (b) there is a saddle mode bifurcation where the cyan blue line crosses the manifold.

Take the space $\{f(x, \mu)\}$ and the set of fs with non-hyperbolic fixed points. This is of co-dimension 1, so typically lines cross it transversely to give saddle node bifurcations; if lines touch the set tangentially but do not cross we get a transcritical bifurcation. In the finite-dimensional case we say that co dimension 1 means dimension one less then the sup-space, but we can have infinite dimensional cases so beware of this.

5.1.1 Summary

In the one dimensional case $\dot{x} = f(x, \mu)$ with a fixed point at 0 when $\mu = 0$, i.e. $f(0, 0) = 0$.

- If $\frac{\partial f}{\partial x}(0, 0) \neq 0$ then the fixed point is hyperbolic and the system is structurally stable.
- If $\frac{\partial f}{\partial x}(0, 0) = 0$ then we have three cases:
  - If $\frac{\partial f}{\partial \mu}(0, 0) \neq 0$ and $\frac{\partial^2 f}{\partial x^2}(0, 0) \neq 0$ then we have a saddle node bifurcation.
  - If $\frac{\partial^2 f}{\partial x^2}(0, 0) = 0$ then the simplest case is the pitchfork bifurcation.
  - If $\frac{\partial f}{\partial \mu}(0, 0) = 0$ then the simplest case is the transcritical bifurcation.

5.2 Bifurcations in $\mathbb{R}^2$ and the Central Manifold Theorem

Theorem (Andronov-Pontryagin)**:** Suppose f is a $\mathcal{C}^1$ vector field defined on a disc $D \subsetneq \mathbb{R}^2$ such that

(I) f, and therefore the flow, points inwards on ∂D.

(II) All fixed points and periodic orbits in D are hyperbolic.

(III) No saddle connections, homoclinic orbits or heteroclinic cycles are in D.

Then f is structurally stable.

Remark: (I) is not so important; it just stops weird things happening at infinity. Usually if you have s system defined on all of $\mathbb{R}^2$ you can make a change of co-ordinates $\underline{y} = \frac{1}{\underline{x}}$ and extend the theorem to all of $\mathbb{R}^2$. The theorem is false in $\mathbb{R}^{n>2}$, although the sense of it remains important.

If a fixed point is non-hyperbolic then Df has an eigenvalue with zero real part:

(I) $\lambda = 0$ when $\mu = 0$ for one real eigenvalue - reduce to the one dimensional case to get saddle nodes et cetera.

(II) $\lambda = \pm i\omega$ when $\mu = 0$ gives a Hopf bifurcation.

(III) λ_1, λ_2 real and both zero: we will not consider this case.

Example 5.2.1: Consider $\dot{x} = \mu - x^2$, $\dot{y} = -y$. We can analyze y and x separately. So we can study separate bifurcations on the x and y-axes using the one dimensional methods.

Example 5.2.2: Consider $\dot{x} = x^2 + xy + y^2$, $\dot{y} = x^2 + xy - y$.

$$\mathrm{D}f = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

This has eigenvalues 0 and -1, with corresponding eigenvectors $(1, 0)^T$ and $(0, 1)^T$ respectively.

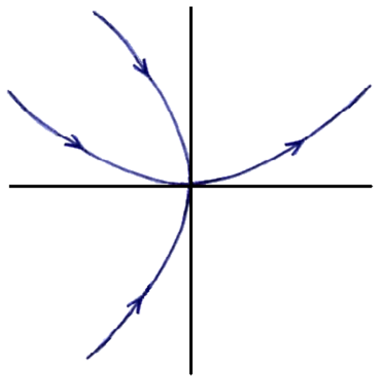

Figure 5.6: The 1-manifold tangent vertical axis (the $\lambda = -1$ direction is the stable manifold, and the 1-manifold tangent to the horizontal axis (the $\lambda = 0$ direction) is the centre manifold.

Does the picture look like this, with the centre manifold and stable manifold tangent to their respective eigenvectors? Yes. We can approximate the centre manifold[1] using the same techniques we used for un/stable manifolds. Assume the CM is given by $y(x) = a_2x^2 + a_3x^3 + \ldots$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2a_2x + 3a_3x^2 + \ldots = \frac{\dot{y}}{\dot{x}} = \frac{x^2 + xy - y}{x^2 + xy + y^2}$$

Equate coefficients to get $y(x) = x^2 - x^3 + \mathrm{h.o.t.}$ But we also need to calculate the motion along the central manifold:

$$\begin{aligned} \dot{x} &= x^2 + xy + y^2 \\ &= x^2 + x(x^2 - x^3 + \ldots) + (x^4 + \ldots) \\ &= x^2 + x^3 + \ldots \end{aligned}$$

So the diagram is right and the fixed point is a saddle node. The calculations so far have been for a particular parameter where the fixed point is non-hyperbolic. How to study the behaviour of a parameterised family?

Example 5.2.3: Look at the three dimensional system $\dot{x} = \mu + x^2 + xy + y^2$, $\dot{y} = x^2 + xy - y$, $\dot{\mu} = 0$. There is a fixed point at $(0, 0, 0)$, and

$$\mathrm{D}f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This has two zero eigenvalues, with eigenspace tangent to the (x, μ)-space, and one eigenvalue -1. Use the same techniques on the extended system of the equation. We should expect a two dimensional centre manifold tangent to the (x, μ)-space and a one dimensional stable manifold tangent to the y-axis. Assume the two dimensional centre manifold is given by $y(x, \mu) = a_{11}x^2 + a_{12}x\mu + a_{22}\mu^2 + \mathrm{h.o.t.}$ Again,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2a_{11}x + a_{12}\mu = \frac{x^2 + xy - y}{\mu + x^2 + xy + y^2}$$

[1]But we still need to show that it exists.

since

$$\dot{y} = \frac{\partial y}{\partial x}\dot{x} + \frac{\partial y}{\partial \mu}\underbrace{\dot{\mu}}_{=0} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}}$$

Compare coefficients as usual to get $y(x, \mu) = x^2 - 2x\mu + 2\mu^2 + \text{h.o.t.}$ Motion on the centre manifold: $\dot{x} = \mu + x^2 + xy + y^2 = \mu + x^2 + \dots$ (substituting in for y), $\dot{\mu} = 0$. We can now forget about varying μ and treat it as a parameter. See saddle node bifurcations.

Theorem (Centre Manifold)**:** Given a non-hyperbolic fixed point in $\mathbb{R}^n$ such that $\mathrm{D}f$ has eigenvectors with zero real part, let E^c be the generalised eigenspace of $\mathrm{D}f$ corresponding to eigenvalues with zero real part. Let

$$E^h := \underbrace{E^{\mathrm{s}}}_{\Re(\lambda)<0} \oplus \underbrace{E^{\mathrm{u}}}_{\Re(\lambda)>0} \qquad \mathbb{R}^n = \underbrace{E^c}_{\Re(\lambda)=0} \oplus \underbrace{E^h}_{\Re(\lambda)\neq 0}$$

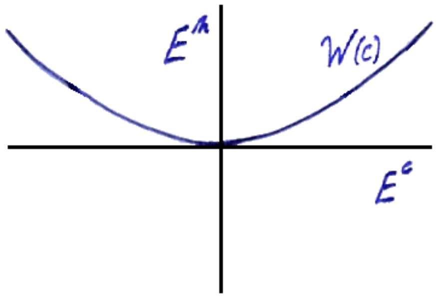

Figure 5.7: The centre manifold $\mathcal{W}(c)$ in blue.

Choose coordinates c in E^c and h in E^h and write the differential equation as

$$\dot{c} = C(c, h) \qquad \dot{h} = H(c, h)$$

(In a previous example we had $E^c = (1, 0)$, $c = x$, $h = y$) Then $\exists\, w : E^c \to E^h$ whose graph $\mathcal{W}^c = \{(c, h) : h = w(c)\}$ is called the **Centre Manifold**. it is tangent to E^c at fixed points, locally invariant under the flow, and such that the dynamics are topologically equivalent to

$$\dot{c} = (c, w(c)) \qquad \dot{h} = h\frac{\partial H}{\partial h}\bigg|_0$$

This is the motion in the centre manifold, written in terms of c (the motion orthogonal to the centre manifold is, up to topological equivalence, given by the linear part of the H equation).

Example 5.2.4: In the last example $E^c = (x, \mu)$, $E^{\mathrm{u}} = (y)$, $w(y) = x^2 - 2x\mu + 2\mu^2$.

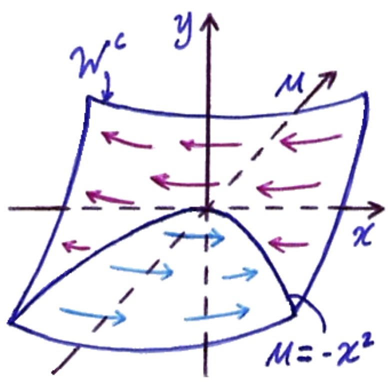

Figure 5.8: Crossing the blue line $\mu = -x^2$ gives saddle node bifurcations.

5.3 Hopf Bifurcations

These occur when the fixed points are non-hyperbolic and the eigenvalues are a complex pair with $\Re(\lambda) = 0$ when $\mu = 0$. The "normal form" is in polar coordinates:

$$(a)\ \begin{array}{l}\dot{r} = \mu r - r^3 \\ \dot{\theta} = 1\end{array} \qquad (b)\ \begin{array}{l}\dot{r} = \mu r + r^3 \\ \dot{\theta} = 1\end{array}$$

Both have a non-hyperbolic fixed point at zero when $\mu = 0$; $\mu > 0 \Rightarrow 0$ is unstable, $\mu < 0 \Rightarrow 0$ stable.

(a) Has a stable focus for $\mu < 0$ and periodic orbits at $r = \sqrt{\mu}$ for $\mu > 0$ and an unstable focus at 0. This is a **Super-Critical** Hopf bifurcation.

(b) Has an unstable focus at 0 and periodic orbits at $r = \sqrt{-\mu}$ for $\mu < 0$ and an unstable focus at 0 for $\mu > 0$. This is a **Sub-Critical** Hopf bifurcation.

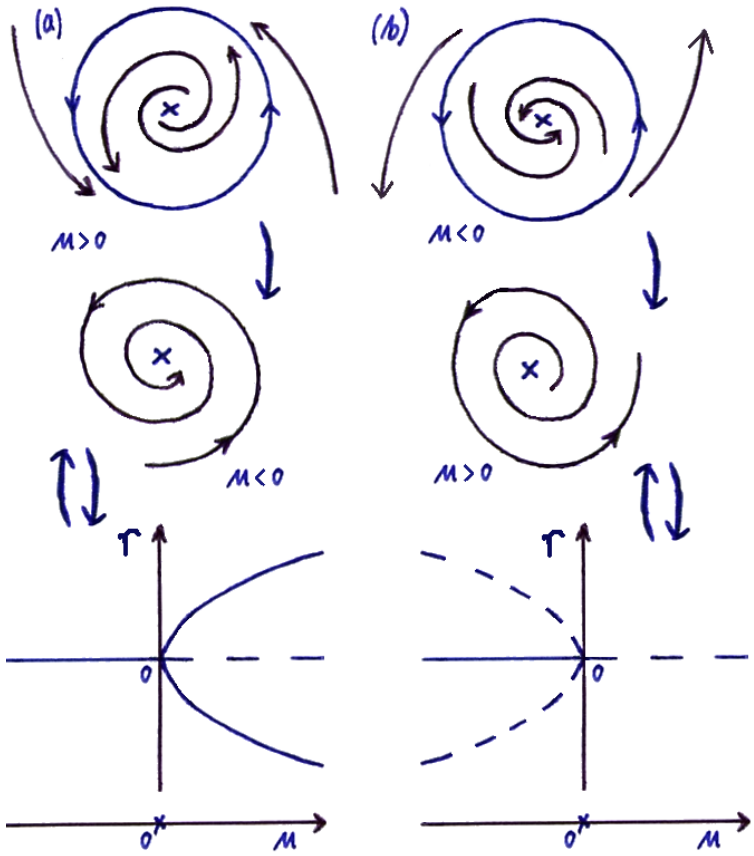

Figure 5.9: (a) A super-critical Hopf bifurcation, with the corresponding pitchfork. (b) a sub-critical Hopf bifurcation.

Theorem (Hopf Bifurcation)**:** For a non-hyperbolic fixed point with a complex pair of eigenvalues with $\Re(\lambda) = 0$ when $\mu = 0$, if $\Re(\lambda)$ passes through zero with non-zero speed as μ passes through zero, and provided the constant $\gamma \neq 0$, then we have a Hopf bifurcation, i.e. a periodic orbit of amplitude approximately $\sqrt{|\mu|}$ and period $\frac{2\pi}{\omega}$, where $\lambda = \pm i\omega$, exists on one or the other side of the bifurcation, depending on the sign of γ.

There are many different versions of this theorem in many different settings. In practice, $\dot{x} = f(x, y, \mu)$, $\dot{y} = g(x, y, \mu)$, with a fixed point at the origin and

$$\mathrm{D}f = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

Calculate the eigenvalues to get a complex pair $\lambda = \pm i\omega$ when $\mu = 0$ and

$$\Re(\lambda) = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) = \mathrm{Tr}(\mathrm{D}f)$$

when $\mu = 0$ we will have $\mathrm{Tr}(\mathrm{D}f) = 0$. The condition about $\Re(\lambda)$ changing at non-zero speed means $\Re(\lambda)$ as a function of μ looks like

$$\left(\frac{\partial^2 f}{\partial x \partial \mu} + \frac{\partial^2 g}{\partial y \partial \mu}\right)\mu$$

and we require

$$\frac{\partial^2 f}{\partial x \partial \mu} + \frac{\partial^2 g}{\partial y \partial \mu} \neq 0$$

So far these are the conditions needed to satisfy the hypotheses of the theorem. If one changes to polar coordinates one gets

$$\dot{r} = \mu r + ?_1 r^2 + ?_2 r^3 + \ldots$$

It turns out that one can get rid of the r^2 term by a change of coordinates, and then $\gamma \neq 0$ is the condition that the r^3 term does not also disappear. γ turns out to be horrific:

$$\begin{aligned}\gamma &= \frac{1}{16}\left(\frac{\partial^3 f}{\partial x^3} + \frac{\partial^3 g}{\partial x^2 \partial y} + \frac{\partial^3 f}{\partial x \partial y^2} + \frac{\partial^3 g}{\partial y^3}\right) \\ &+ \frac{1}{16}\left(\frac{\partial f^4}{\partial x^3 \partial y} + \frac{\partial f^4}{\partial x \partial y^3} - \frac{\partial g}{\partial x^3 \partial y} - \frac{\partial g^4}{\partial x \partial y^3} - \frac{\partial f^2}{\partial x^2}\frac{\partial g^2}{\partial x^2} + \frac{\partial f^2}{\partial y^2}\frac{\partial g^2}{\partial y^2}\right)\end{aligned}$$

Example 5.3.1: In the Lorenz equations with $r > 1$ and non-zero fixed points it is easy to show that there is a Hopf bifurcation for some r, but it is quite complicated to see if it is super-critical or sub-critical.

5.4 Co-Dimension 1 Bifurcations of Periodic Orbits

Recall that we analysed behavior near periodic orbits by looking at the return map Φ on a small section Σ transverse to the periodic orbit. The critical thing here is the Floquet multipliers $\{\mu_1, \ldots, \mu_n\}$. On the periodic orbit, $y^* = \Phi(y^*)$, and in $\Sigma \setminus \{y^*\}$ we have

$$\Phi : y \mapsto y^* + A(y - y^*) + \mathrm{h.o.t.}$$

So the μ_i are the eigenvalues of A, so the periodic orbit is non-hyperbolic in the case where $|\mu_i| = 1$ for $i \in \{1, \ldots, n\}$. Co-dimension one bifurcations occur as you change a parameter α in such a way that a μ_i goes through $|\mu_i| = 1$. There are three cases:

(I) $\mu_i \in \mathbb{R}$, passes through 1.

(II) $\mu_i \in \mathbb{R}$, passes through -1.

(III) A pair $\mu_i, \mu_j \in \mathbb{C}$ such that $\mu_i = \overline{\mu_j}$ and have $|\mu_i| = |\mu_j| = 1$.

(I) is a saddle node bifurcation of periodic orbits.[2] Consider

$$\dot{r} = r - \alpha r^2 + r^3$$

Vary α: $\dot{r} = 0$ when $r = 0$ or when $r^2 - \alpha r + 1 = 0$; when $\alpha < 2$ there are no solutions, when $\alpha = 2$ there is one and when $\alpha > 2$ there are two solutions. (I) is the only situation that can happen to periodic orbits in $\mathbb{R}^2$, as there is only one Floquet multiplier, ruling out (III), and flows cannot cross the periodic orbit in $\mathbb{R}^2$, ruling out (II). Type(II) is a period-doubling bifurcation, which is co-dimension 1 and seen in many examples (see figure 5.11). But in $\mathbb{R}^{n \geq 3}$, types (II) and (III) both occur.
In $\mathbb{R}^3$ (III) gives a bifurcation of an invariant torus. This has co-dimension 1, but resonances between frequencies mean that the actual behaviour is very complicated

[2]See question 5 of example sheet 4.

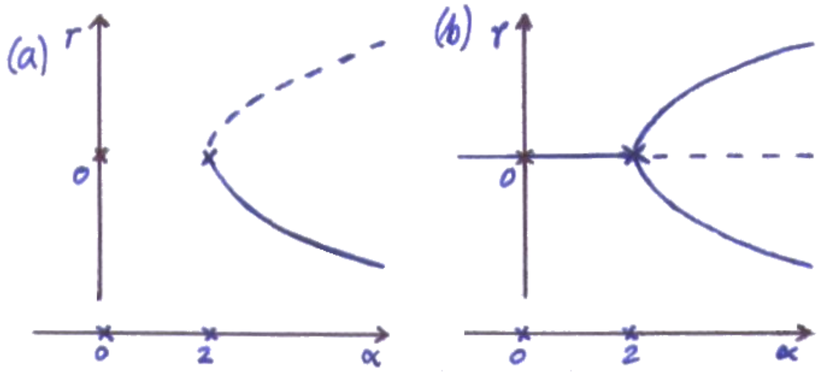

Figure 5.10: (a) The modified saddle node. (b) The modified pitchfork.

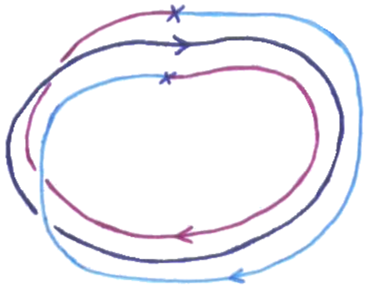

Figure 5.11: In $\mathbb{R}^3$, as μ varies through -1 period doubling bifurcations are observed.

5.5 Global Bifurcations

Consider the forced pendulum $\dot{\theta} = p$, $\dot{p} = F - kp - \sin(\theta)$, where F and k are two non-negative parameters.

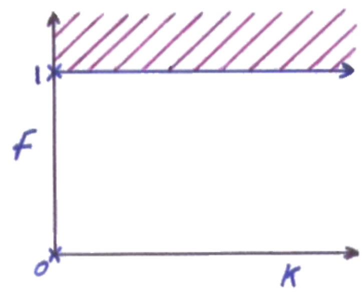

Figure 5.12: There are no fixed points for $F > 1$, one (non-hyperbolic) fixed point at $F = 1$ and two fixed points for $0 \leq F < 1$. At $F = 0$ there are no periodic orbits.

Guess that at $F = 1$ there is a non-hyperbolic fixed point, and as F passes through 1 there is a saddle node bifurcation of fixed points: $F < 1 \Rightarrow \exists$ two fixed points (one stable, one saddle), and $F > 1 \Rightarrow \nexists$ fixed points.[3] What bifurcations occur that involve the periodic orbit?

(I) This is not a Hopf bifurcation, as analysis of the fixed points shows that the conditions of the Hopf theorem are not satisfied, and we know that the orbit encircles the cylinder, and so cannot arise in a Hopf bifurcation.

(II) It is also clearly not a saddle node bifurcation as we do not go from 0 to 2 periodic orbits when F passes through 1.

[3]It is very worthwhile to check that you can do this through calculations and that the diagram is complete.

The answer is that there is a unique periodic orbit in the shaded region and none in the unshaded region.

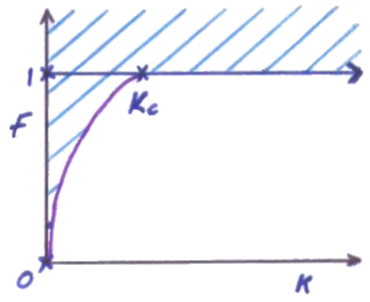

Figure 5.13: Passing through the purple line from (0,0) to $(k_c, 1)$ gives a global homoclinic bifurcation, while passing through the blue line for $k > k_c$ gives "saddle node on a cycle" bifurcations.

Crossing the line $(1, k > k_c)$ gives a "**Saddle Node on a Cycle Bifurcation**" while crossing into the shaded region for $k \in [0, k_c)$ gives a "**Global Homoclinic Bifurcation**".

5.5.1 Saddle Node on a cycle

Consider the following diagram:

Figure 5.14: In the last stage of the bifurcation, the flow is slow near where the fixed point used to be and fast on the opposite side of the periodic orbit.

Considering only the fixed points this is a standard saddle node bifurcation, but globally the destruction of the fixed points coincides with the creation of a periodic orbit. This is what occurs as F passes through 1 for $k > k_c$. A proof would be hard (but just about doable), but the point is that this is plausible - it is the only thing that fits with the theory, and numerical experiments on the equations "confirms" this. In order to get this "proof", you would need to do numerical experiments on a computer. Now we want to show the existence, for small positive k and F, of a homoclinic orbit Γ. Consider $H = \frac{1}{2}p^2 - \cos\theta$. For F, k small, can we find Γ homoclinic? If so then

$$\int_\Gamma \dot{H} \mathrm{d}t = 0$$

We shall suppose that Γ is close to γ^+ in order to compute the integral approximately (similar to the averaging method or the energy balance method used on the Van de Pol equations for small

β). This integral is known as the **Melnikov Integral** in this context (homoclinic).

$$\begin{aligned}\int_\Gamma \dot{H}\,\mathrm{d}t &= \int_\Gamma \left(pF - kp^2\right)\mathrm{d}t \\ &\approx \int_{-\pi}^{\pi} \frac{\left(pF - kp^2\right)}{\dot{\theta}(=p)}\mathrm{d}\theta \\ &= \int_{-\pi}^{\pi} (F - kp)\,\mathrm{d}\theta \\ &= \int_{-\pi}^{\pi} \left(F - 2k\cos\left(\frac{\theta}{2}\right)\right)\mathrm{d}\theta \\ &= 2\pi F - 8k\end{aligned}$$

So there is a homoclinic orbit for $F = \frac{4k}{\pi}$. To compute the rest of the curve (i.e. for larger F and k) and the point k_c you will need the Computer Scientist again.

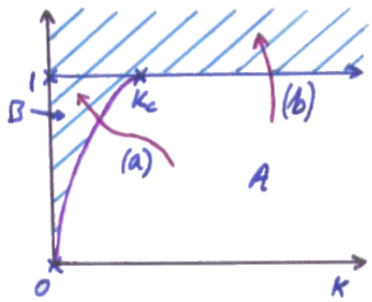

Figure 5.15: See the next two figures for the bifurcations given by cases (a) and (b).

Varying the parameters to get from A to B:

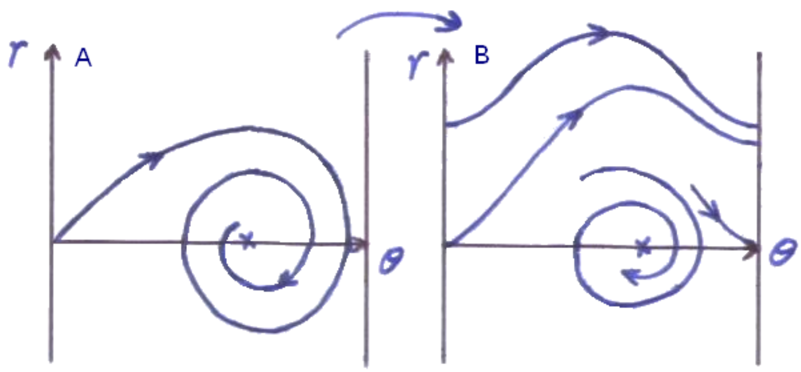

Figure 5.16: Case (a): A stable periodic orbit is created.

B has both a stable fixed point (the pendulum hanging down) and a stable periodic orbit. For varying k along $F = 1$, see figure 5.17.

5.5.2 Homoclinic Orbits

It remains to understand what happens on homoclinic orbits. Assume we have a system that behaves similarly to figure 5.2 as a parameter μ is varied through 0. Draw a box of size $\pm\varepsilon$ ($\varepsilon > 0$ obviously). Label the top Σ_y and the right hand side Σ_x, then follow 4 steps:

Step (1) Take a point $(x_0, \varepsilon) \in \Sigma_y$ with $x_0 > 0$.

Step (2) Follow the orbit through the box to a point $(\varepsilon, y_0) \in \Sigma_x$.

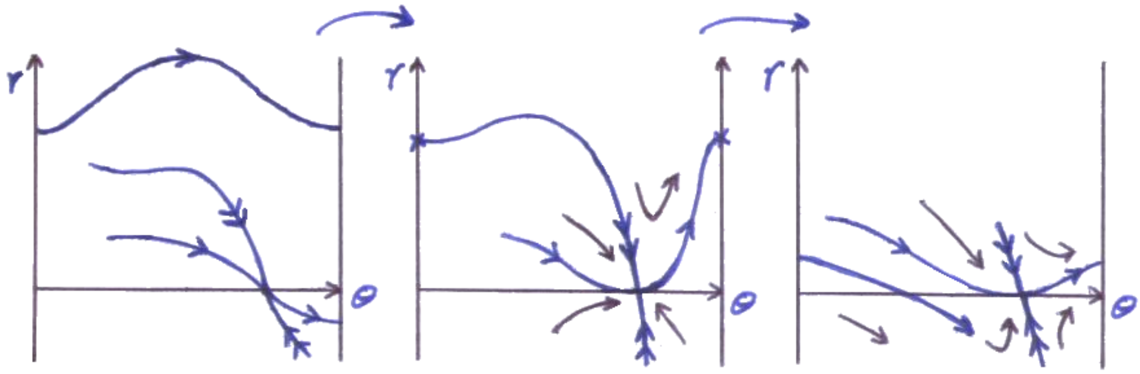

Figure 5.17: Case (b): The stable periodic orbit becomes a homoclinic orbit before being lost.

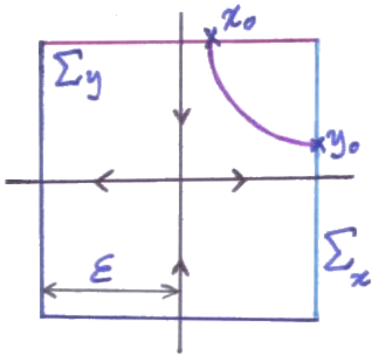

Figure 5.18: Steps one and two.

Step (3) Follow the orbit from (ε, y_0) back to Σ_y close to the unstable manifold of the fixed point.

Step (4) Get back to Σ_y at (x_1, ε). Look for solutions to $x_0 = x_1 \Rightarrow$ periodic orbit.

The point is that inside the box the flow travels slowly and is thus dominated by the linear terms $\dot{x} = \lambda_1 x$, $\dot{y} = -\lambda_2 y$ with λ_1, $\lambda_2 > 0$ (for simplification assume that these are independent of μ). Outside of the box the flow travels relatively fast and depends on μ.

Step (2) The time T spent in the box is given by $\varepsilon = x_0 e^{\lambda_1 T} \Rightarrow T = \frac{1}{\lambda_1} \log\left(\frac{\varepsilon}{x_0}\right)$, so $y_0 = \varepsilon e^{-\lambda_2 T} \Rightarrow$ $y = \text{const} x_0^{\lambda_2/\lambda_1}$.

Step (3) $x_1 = \text{const}_1 \mu + \text{const}_2 y_0 = \mu + x_0^{\lambda_1/\lambda_2}$ (ignoring the constants (!)).

Step (4) Look for solutions to $x = \mu + x_0^{\lambda_2/\lambda_1}$ in $x > 0$. If $\frac{\lambda_2}{\lambda_1} > 1$, then

- $\mu < 0 \Rightarrow$ no solutions
- $\mu = 0 \Rightarrow$ one solution at $x = 0$ (the homoclinic orbit)
- $\mu > 0 \Rightarrow$ one solution (stable periodic orbit)

If $\frac{\lambda_2}{\lambda_1} < 1$ then

- $\mu < 0 \Rightarrow$ one solution (unstable periodic orbit)
- $\mu = 0 \Rightarrow$ one solution at $x = 0$ (the homoclinic orbit)
- $\mu > 0 \Rightarrow$ no solutions

So there are two possibilities depending on whether $\lambda_1 < \lambda_2$ (contraction) or whether $\lambda_1 > \lambda_2$ (expansion).

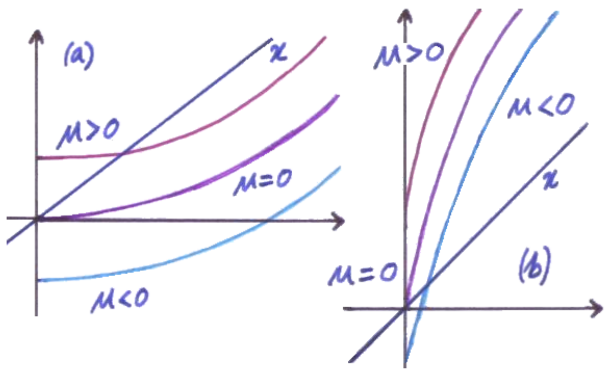

Figure 5.19: The number of intersections with the dark blue line gives the gives the number of solutions. (a) when $\frac{\lambda_2}{\lambda_1} > 1$. (b) When $\frac{\lambda_2}{\lambda_1} < 1$.

Exercise: Check that for the damped pendulum the eigenvalues at the saddle satisfy $\frac{\lambda_2}{\lambda_1} > 1 \Rightarrow$ stable periodic orbit created in the homoclinic bifurcation.