Definability equals recognizability for graphs of bounded treewidth

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How to construct graphs

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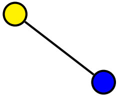

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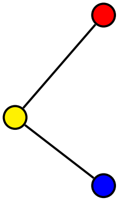

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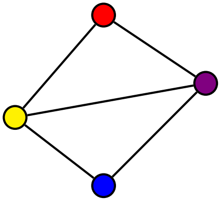

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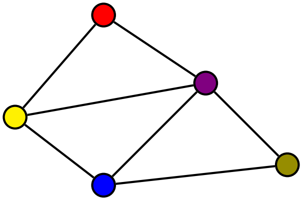

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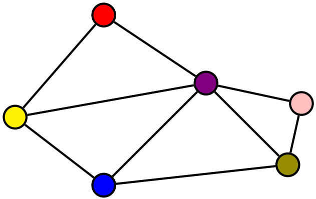

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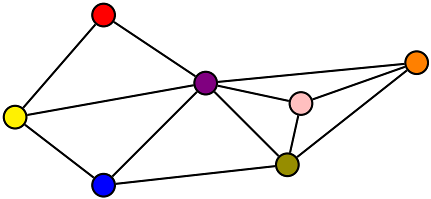

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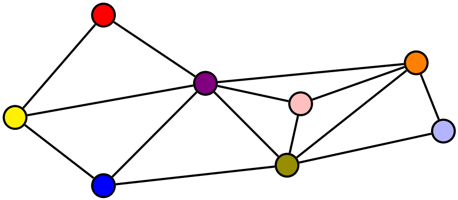

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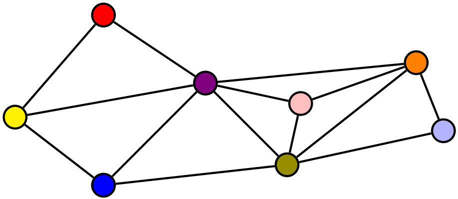

Idea: Keep only a small number of vertices in memory.

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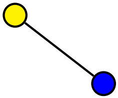

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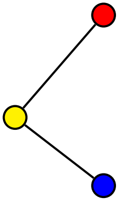

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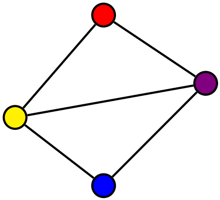

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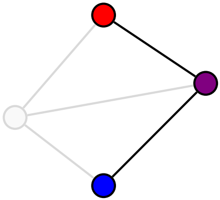

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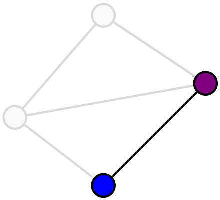

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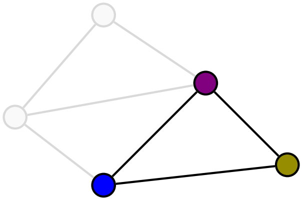

Introduce a new active vertex

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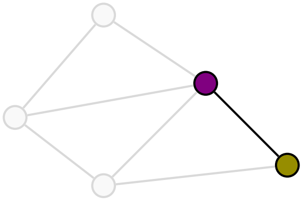

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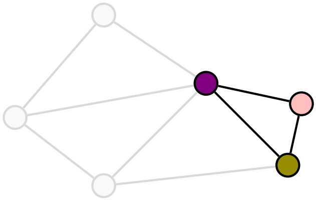

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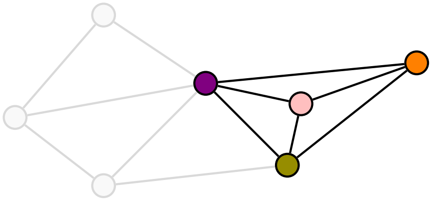

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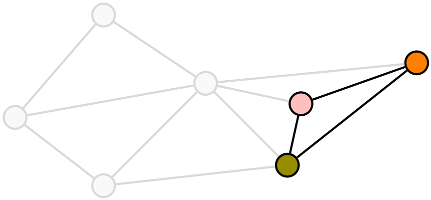

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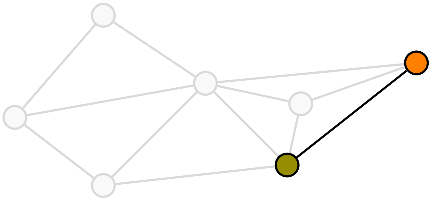

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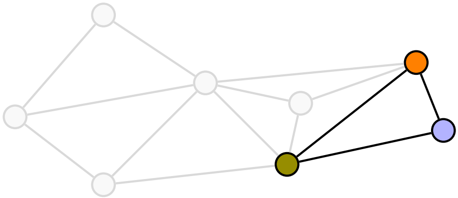

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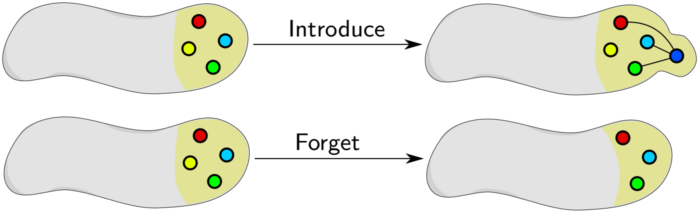

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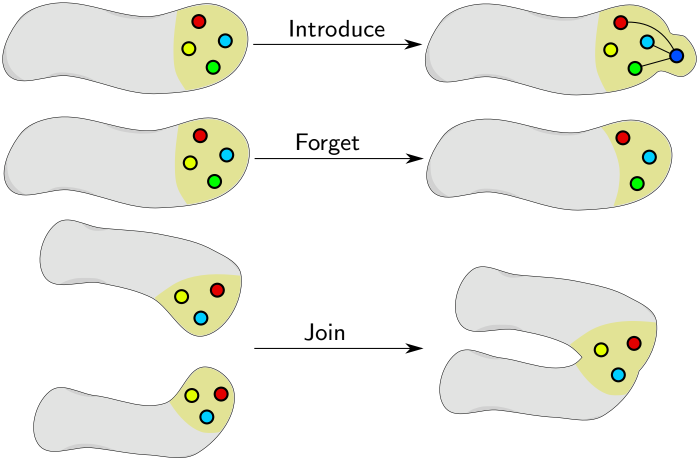

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  - The parameter k is the *width* of the decomposition.

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- **Courcelle's conjecture**: If Π can be verified by an automaton on a tree decomposition, then Π is expressible in MSO.

Recognizability

- Graph property Π is k-**recognizable** if the following Myhill-Nerode relation $\equiv_k$ over k-interface graphs has finite index.

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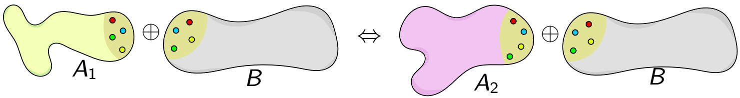

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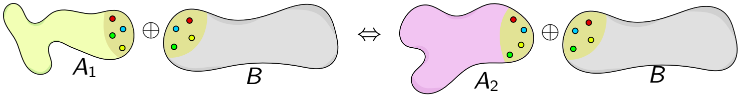

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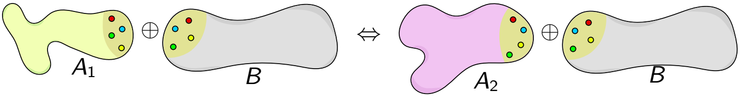

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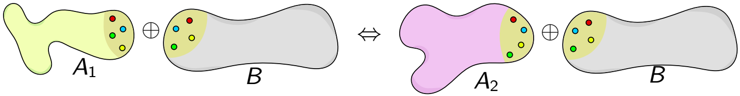

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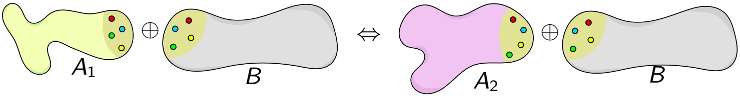

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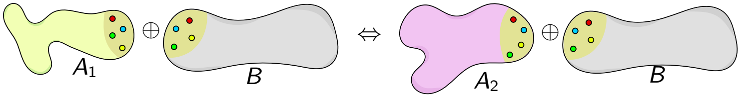

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  - **WRONG** for multiple reasons.

Courcelle's conjecture

Courcelle's conjecture Courcelle; ∼'90

Suppose

- Π is a recognizable graph property, and
- $\mathcal{T}_k$ is the class of graphs of treewidth at most k, for some constant k.

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Theorem — Bojańczyk, P.; 2016

Courcelle's conjecture holds.

Attempt on the proof

- By the finiteness of the Myhill-Nerode equivalence relation, there is a tree automaton $\mathcal{A}$ that:
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- Everything boils down to "defining" in MSO some tree decomposition of bounded width.

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Main theorem

There is a (nondeterministic) MSO transduction that, given a graph of treewidth k, outputs its tree decomposition of width at most $f(k)$, for some function f.

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- **Now**: A combinatorial notion of an "MSO-definable" decomposition.

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A *guidance system* Λ in a graph G is a set of rooted forests

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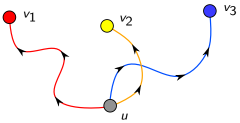
where v_i is the root of the tree of F_i that contains u.



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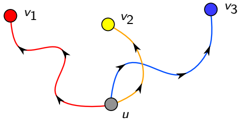

- A vertex subset X is *captured* by Λ if $X \subseteq \Lambda(u)$ for some vertex u.

Capturing tree decompositions

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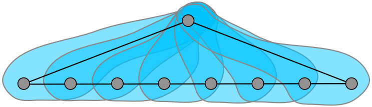

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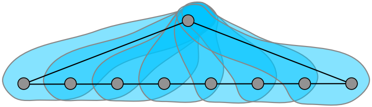

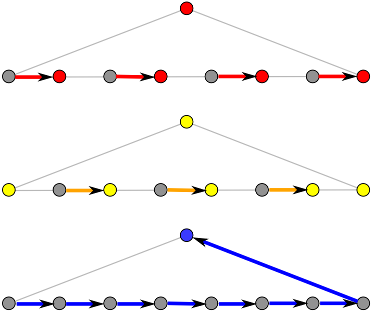

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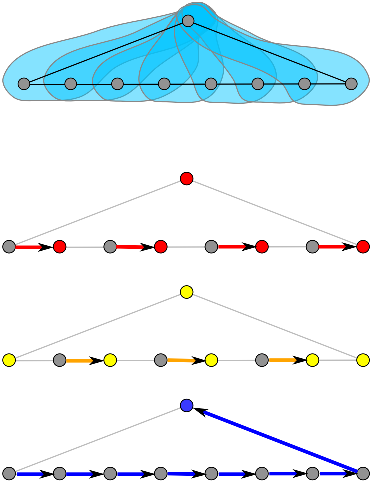

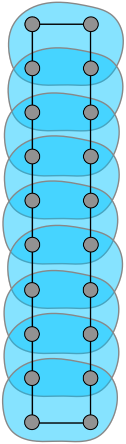

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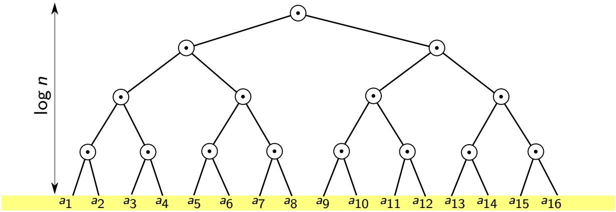

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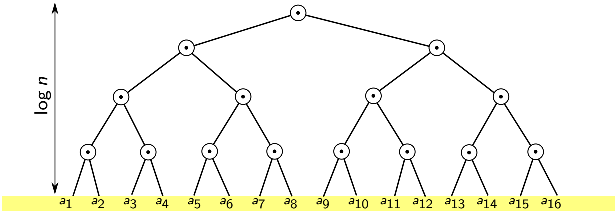

- We need **constant** depth, depending only on $|S|$.

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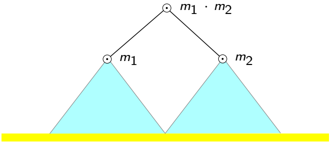

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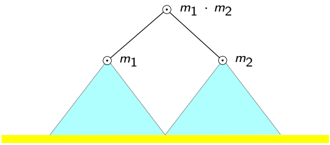

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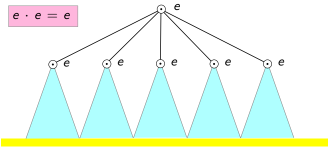

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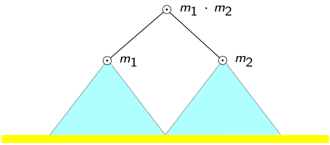

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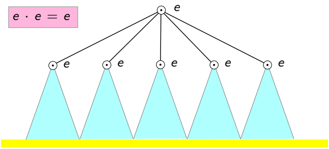

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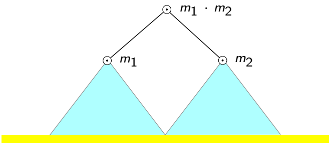

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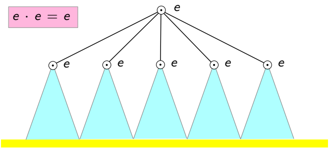

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path decomp. of width k $\Rightarrow$ word over a semigroup of size $f(k)$

apply induction on the depth of the factorization forest

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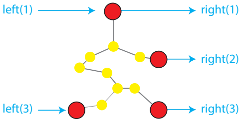

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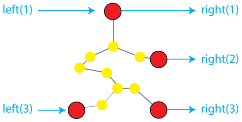

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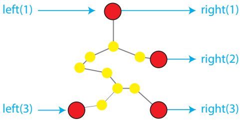

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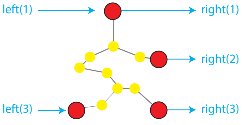

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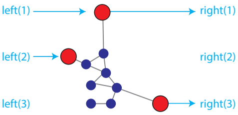

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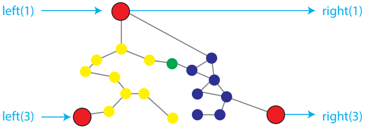

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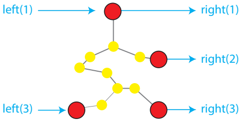

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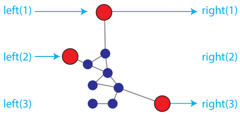

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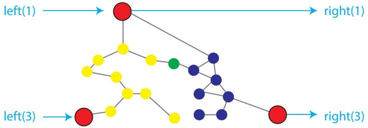

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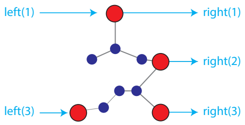

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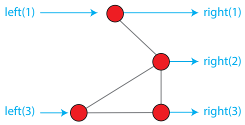

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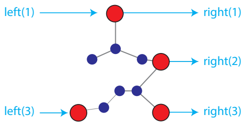

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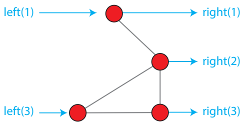

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- This forms a semigroup $\mathcal{S}$ of size $2^{\mathcal{O}(k^2)}$.

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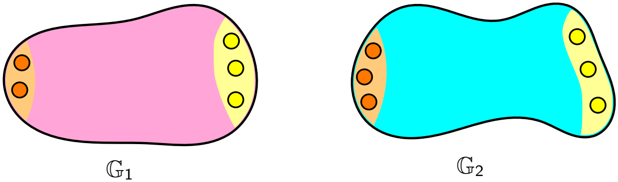

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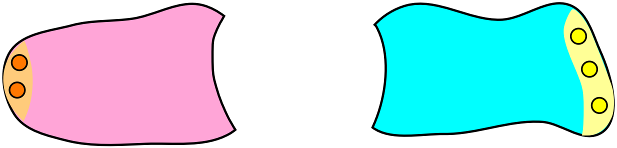

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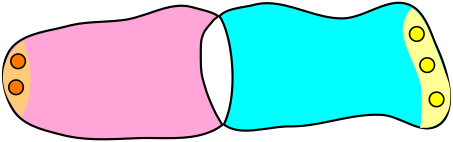

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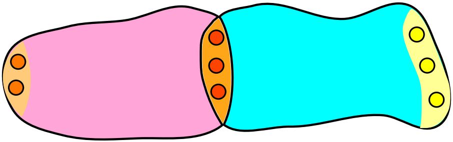

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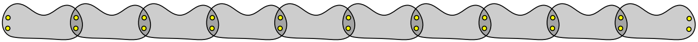

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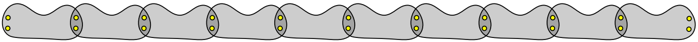

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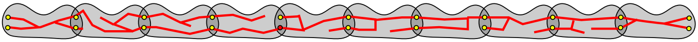

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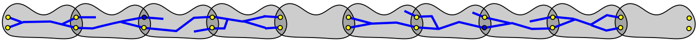

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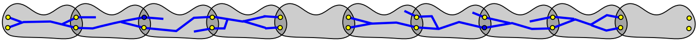

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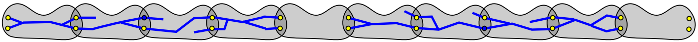

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- **Thanks for attention!**