Structural Limits

Jaroslav Nešetřil Patrice Ossona de Mendez

Charles University
Praha, Czech Republic

LIA STRUCO

CAMS, CNRS/EHESS
Paris, France

— Warwick $\binom{2^{\binom{2^2}{2}}}{2}$ —

Contents

Introduction

Issues

- How to describe/approximate a network?
- How much is a network structured? How much is it random-like?
- How to check whether a network has (or is close to have) some property?
- How to compare the structures of two networks?
- How to represent limits of networks?
- Asymptotic structure of the networks in a convergent sequence?

Structural Limits

Structural Limits

Definition (Stone pairing)

Let ϕ be a first-order formula with p free variables and let $G = (V, E)$ be a graph.

The *Stone pairing* of ϕ and G is

$$\langle \phi, G \rangle = \Pr(G \models \phi(X_1, \dots, X_p)),$$

for independently and uniformly distributed $X_i \in G$.

That is:

$$\langle \phi, G \rangle = \frac{|\phi(G)|}{|G|^p}.$$

Structural Limits

Definition

A sequence (G_n) is *X-convergent* if, for every $\phi \in X$, the sequence $\langle \phi, G_1 \rangle, \ldots, \langle \phi, G_n \rangle, \ldots$ is convergent.

FO_0	Sentences	Elementary limits
QF	Quantifier free formulas	Left limits
$\mathrm{FO}^{\mathrm{local}}$	Local formulas	Local limits
FO	All first-order formulas	FO-limits

General Representation Theorems

Three Types of Limits Objects

	Non-Standard	Distributional	Analytic
Dense (Left limit)	Ultraproduct + Loeb measure (Elek, Szegedy '07)	Exchangeable random graph (Aldous '81, Hoover '79)	Graphon (Lovász *et al.* '06)
Sparse (Local limit)	—	Unimodular distribution (Benjamini, Schramm '01)	Graphing (Elek '07)
General (Structural limit)	Ultraproduct + Loeb measure (Nešetril, POM '12)	Invariant distribution (Nešetril, POM '12)	

Non-Standard Limit: Ultraproduct with Loeb Measure

Theorem (Nešetril, POM 2012)

Let $(G_n)_{n\in\mathbb{N}}$ be FO-convergent and let U be a non-principal ultrafilter on $\mathbb{N}$. Then there exists a probability measure ν on the ultraproduct $\prod_U G_n$ such that for every first-order formula ϕ with p free variables it holds:

$$\int_{(\prod_U G_n)^p}\cdots\int \mathbf{1}_\phi([x_1],\dots,[x_p])\ \mathrm{d}\nu([x_1])\ \dots\ \mathrm{d}\nu([x_p]) = \lim_U\langle\psi, G_i\rangle.$$

— **Not product σ-algebra, but Fubini-like properties** —

(Follows Elek, Szegedy '07; See also Keisler '77)

Distributionual Limit

Theorem (Nešetřil, POM 2012)

There are maps $G \mapsto \mu_G$ and $\phi \mapsto k(\phi)$, such that

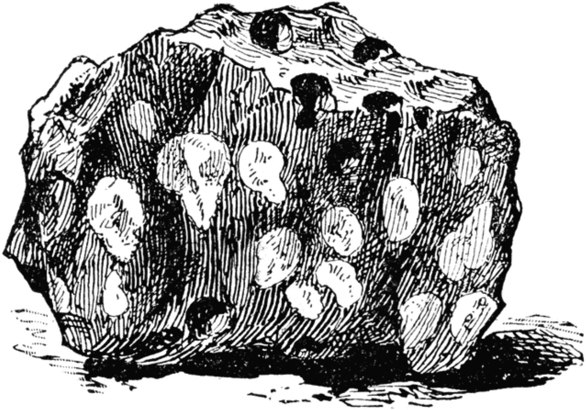
- $G \mapsto \mu_G$ is injective
- $\langle \phi, G \rangle = \int_S k(\phi)\,\mathrm{d}\mu_G$
- A sequence $(G_n)_{n\in\mathbb{N}}$ is X-convergent iff μ_{G_n} converges weakly.

Thus if $\mu_{G_n} \Rightarrow \mu$, it holds

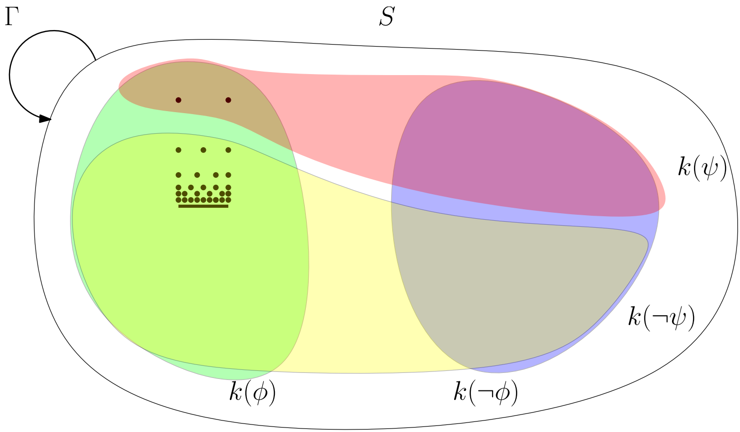
$$\int_S k(\phi)\,\mathrm{d}\mu = \lim_{n\to\infty} \int_S k(\phi)\,\mathrm{d}\mu_{G_n} = \lim_{n\to\infty} \langle \phi, G_n \rangle.$$

Note: $\mathrm{FO}_p \to \mathfrak{S}_p$-invariance; $\mathrm{FO} \to \mathfrak{S}_\omega$-invariance.

Stone Spaces

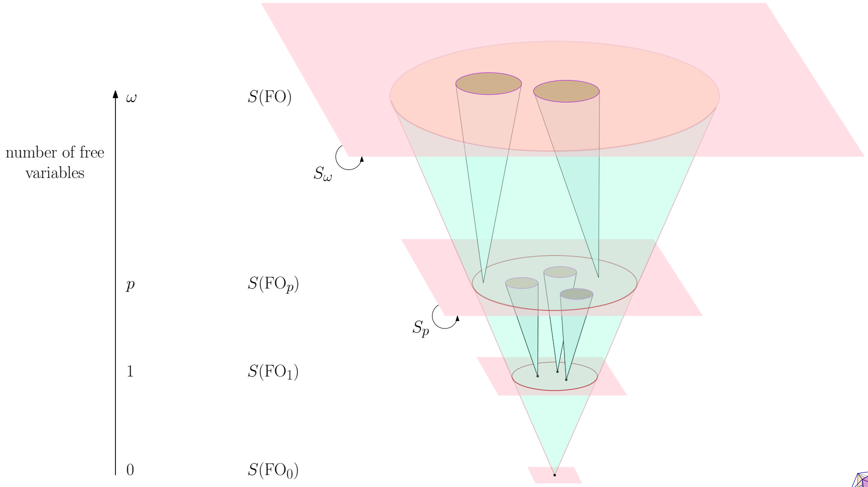


Stone Space



A topological version of Venn diagrams

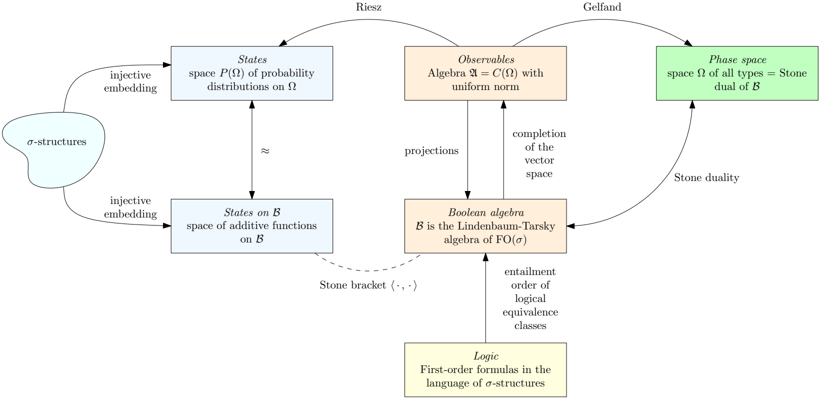
Stone Spaces



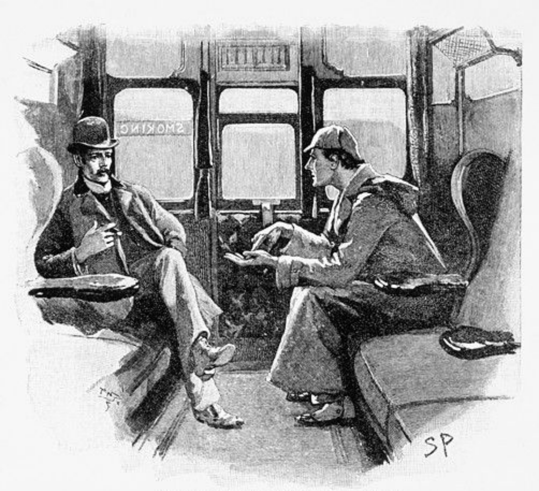
Structural Limits

Boolean algebra $\mathcal{B}(X)$	Stone Space $S(\mathcal{B}(X))$
Formula ϕ	Continuous function f_ϕ
Vertex v	"Type of vertex" T
Structure $\mathbf{A}$	probability measure $\mu_{\mathbf{A}}$
$\langle \phi, \mathbf{A} \rangle$	$\int f_\phi(T)\ \mathrm{d}\mu_{\mathbf{A}}(T)$
X-convergent $(\mathbf{A}_n)$	weakly convergent $\mu_{\mathbf{A}_n}$
$\Gamma = \mathrm{Aut}(\mathcal{B}(X))$	Γ-invariant measure

Ingredients of the proof



The Elementary Convergence Case



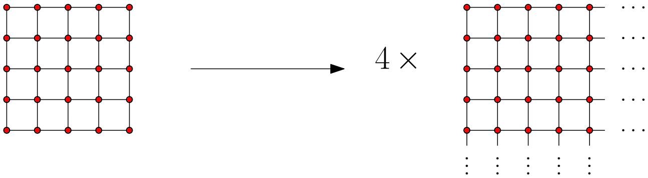
"HOLMES GAVE ME A SKETCH OF THE EVENTS."

Elementary convergence

For $\phi \in \mathrm{FO}_0$, we have

$$\langle \phi, G\rangle = \begin{cases} 1 & \text{if } G \models \phi, \\ 0 & \text{otherwise.} \end{cases}$$

FO_0-convergence is called elementary convergence.



Limit Object

Proposition (Gödel+Löwenheim–Skolem)

Every elementarily convergent sequence of finite graphs has a limit, which is an at most countable graph.

Complete theories with Finite Model Property form a closed subset of the Stone dual of FO_0 but ...

No characterization of elementary limits

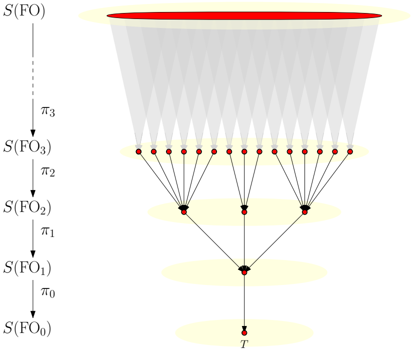
Trakhtenbrot's theorem states that the problem of existence of a finite model for a single first-order sentence is undecidable.

Special Elementary Limits 1: ω-categorical

A complete theory T is *ω-categorical* if it has a unique countable model.

$\iff \forall p \in \mathbb{N}$, the Stone dual of FO_p/T is finite

$\iff$ every countable model G of T has an oligomorphic automorphism group: $\forall n \in \mathbb{N}$, G^n has finitely many orbits under the action of $\mathrm{Aut}(G)$.



Special Elementary Limits 2: Ultrahomogeneous

A graph G is *ultrahomogeneous* if every isomorphism between two of its induced subgraphs can be extended to an automorphism. The only countably infinite homogeneous graphs are:

- ωK_n, nK_ω, ωK_ω, and complements;
- the Rado graph;
- the Henson graphs and complements.

Proposition

If $(G_n)_{n\in\mathbb{N}}$ is elementarily convergent to an ultrahomogeneous graph, then $(G_n)_{n\in\mathbb{N}}$ is FO-convergent if and only if $(G_n)_{n\in\mathbb{N}}$ is QF-convergent.

Example

Theorem (Nešetril, Ossona de Mendez)

Let $0 < p < 1$ and let $G_n \in \mathrm{G}(n,p)$ be independent random graphs with edge probability p. Then $(G_n)_{n\in\mathbb{N}}$ is almost surely FO-convergent.

Proof.

$(G_n)_{n\in\mathbb{N}}$ almost surely converges elementarily to the Rado graph, and almost surely QF-converges. □

Problem (Cherlin)

Is the generic countable triangle-free graph elementary limit of finite graphs?

The Quantifier-Free Case

Left Convergence

$$F \quad \mapsto \quad \phi_F = \bigwedge_{ij \in E(F)} (x_i \sim x_j)$$

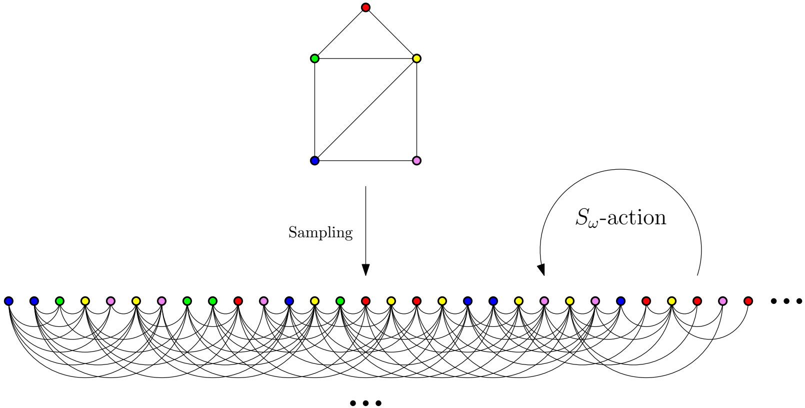
Then

$$\langle \phi_F, G \rangle = \frac{\hom(F,G)}{|G|^{|F|}} = t(F,G).$$

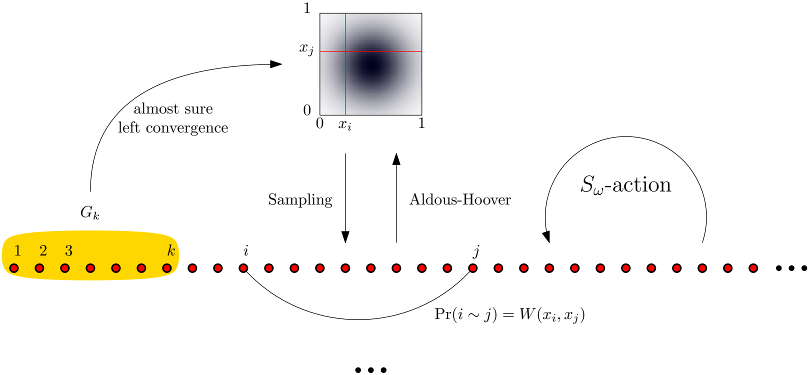
Hence, if $|G_n| \to \infty$

$(G_n)_{n\in\mathbb{N}}$ is left convergent if and only if it is QF-convergent.

The Infinite Exchangeable Graph



The Infinite Exchangeable Graph



Extensions

- ✔ colored, directed, decorated graphs (Lovász, Szegedy '10);
- ✔ regular hypergraphs (Elek, Szegedy '12; Zhao '14);
- ✔ relational structures (Aroskar '12; Aroskar, Cummings '14);
- ☛ algebraic structures.

Algebraic Structures

Signature $\sigma = (f_0, \ldots, f_d)$, f_i involution

$\longrightarrow$ encodes graphs with maximum degree d;

$\longrightarrow$ QF_1-limit equivalent to local limit;

$\longrightarrow$ limit object with same signature, f_i measure preserving involution (= graphing).

Thus...

General QF-convergence extends both left limits and local limits of graphs with bounded degrees.

The Local Case

Local Formulas

Definition

A formula ϕ is *local* if there exists r such that satisfaction of ϕ only depends on the r-neighborhood of the free variables:

$$G \models \phi(v_1, \dots, v_p) \iff G[N_r(\{v_1, \dots, v_p\})] \models \phi(v_1, \dots, v_p).$$

Definition

A sequence (G_n) is *local-convergent* if, for every $\phi \in \mathrm{FO}^{\mathrm{local}}$, the sequence $\langle \phi, G_1 \rangle, \dots, \langle \phi, G_n \rangle, \dots$ is convergent.

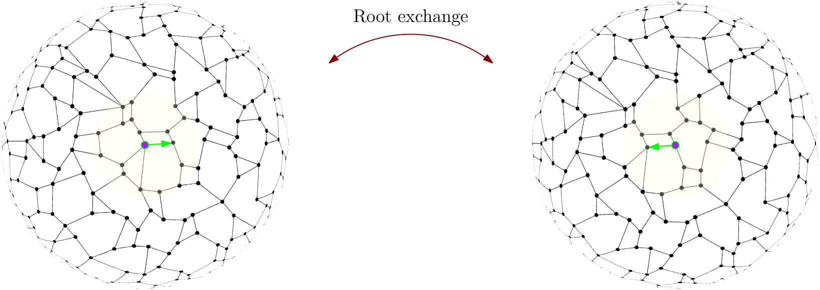
> (G_n) is local-convergent if, for every local formula ϕ, the probability that G_n satisfies ϕ for a random assignment of the free variables converges.

Local Convergent Sequence of Bounded Degree Graphs

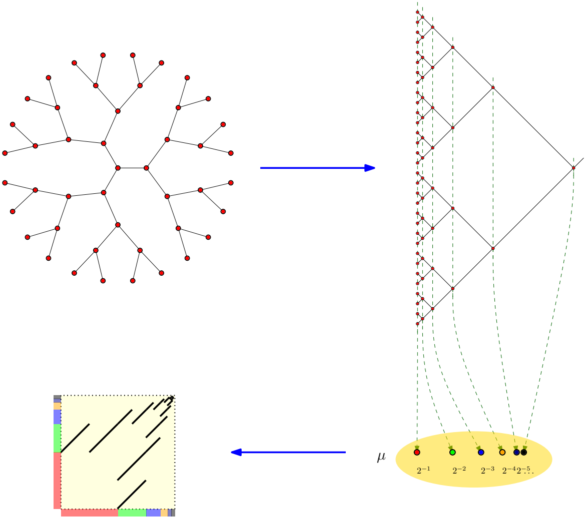
For a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs with degree $\leq d$ the following are equivalent:

1. the sequence $(G_n)_{n\in\mathbb{N}}$ is local convergent (in the sense of Benjamini and Schramm);
2. the sequence $(G_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}_1^{\mathrm{local}}$-convergent;
3. the sequence $(G_n)_{n\in\mathbb{N}}$ is local-convergent (in our sense).

The Unimodular Distribution



Example



Why Formulas?

Consider extension of local convergence: $(G_n)_{n\in\mathbb{N}}$ converges if, for every d and rooted (F,r) there is some $t_d(F)$ such that

$$\Pr[B_d(G_n,X)\simeq(F,r)]\longrightarrow t_d(F).$$

Why Formulas?

Consider extension of local convergence: $(G_n)_{n\in\mathbb{N}}$ converges if, for every d and rooted (F,r) there is some $t_d(F)$ such that

$$\Pr[B_d(G_n,X)\simeq(F,r)]\longrightarrow t_d(F).$$

No limit probability distribution!

Example: G_n any n-regular graph. Then for every d and every (F,r) it holds

$$\Pr[B_d(G_n,X)\simeq(F,r)]\longrightarrow 0.$$

Why Local Convergence?

Proposition (Nešetřil, Ossona de Mendez)

A sequence $G_1, \dots, G_n, \dots$ of graphs is FO-convergent if and only if it is both local convergent and elementarily convergent.

Theorem (Gaifman)

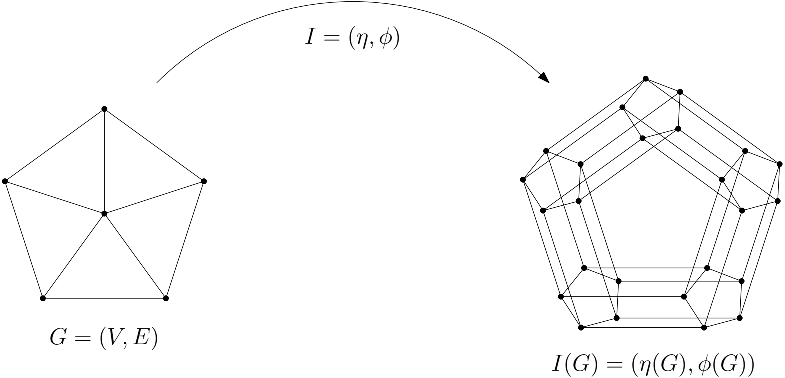
Every formula ϕ is equivalent to a Boolean combination of local formulas and sentences of the form

$$\exists y_1 \dots \exists y_m \left(\bigwedge_{1 \leq i < j \leq m} \mathrm{dist}(y_i, y_j) > 2r \wedge \bigwedge_{1 \leq i \leq m} \psi(y_i) \right)$$

where ψ is local.

Interpretations

Interpretation



$$\eta(x_1, x_2) := (\deg(x_1) = 3) \wedge (\deg(x_2) = 3)$$

$$\phi(x_1, x_2; y_1, y_2) := ((x_1 \sim y_1) \wedge (x_2 = y_2)) \vee ((x_1 = y_1) \wedge (x_2 \sim y_2))$$

Basic Properties

Every interpretation I of σ'-structures in σ-structures define

- a mapping $\mathbf{A} \mapsto \mathsf{I}(\mathbf{A})$ from $\mathrm{Rel}(\sigma)$ to $\mathrm{Rel}(\sigma')$
- a mapping $\phi \mapsto \mathsf{I}(\phi)$ from $\mathrm{FO}(\sigma')$ to $\mathrm{FO}(\sigma)$

such that for every $\mathbf{v}_1, \dots, \mathbf{v}_p$ it holds

$$\mathsf{I}(\mathbf{A}) \models \phi(\mathbf{v}_1, \dots, \mathbf{v}_p) \quad \iff \mathbf{A} \models \mathsf{I}(\phi)(\mathbf{v}_1, \dots, \mathbf{v}_p).$$

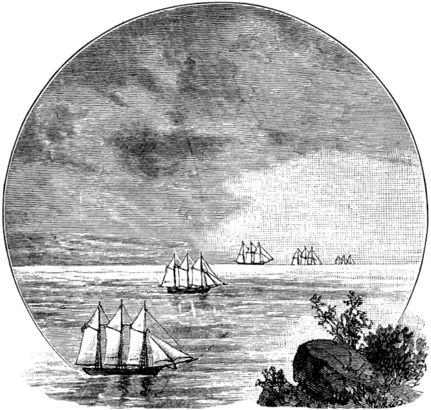
In other words:

$$\phi(\mathsf{I}(\mathbf{A})) = \mathsf{I}(\phi)(\mathbf{A}).$$

Thus if the domain of $\mathsf{I}(\mathbf{A})$ is $\eta(\mathbf{A})$ and if ϕ has p free variables it holds

$$\langle \phi, \mathsf{I}(\mathbf{A}) \rangle = \frac{\langle \mathsf{I}(\phi), \mathbf{A} \rangle}{\langle \eta, \mathbf{A} \rangle^p}$$

Near the Limit



Negligible Sequences

Definition

Let $\mathsf{G} = (G_n)_{n\in\mathbb{N}}$ be a local-convergent sequence. A sequence $\mathsf{X} = (X_n)_{n\in\mathbb{N}}$ of subsets $X_n \subseteq V(G_n)$ is *negligible* and we note $\mathsf{X} \approx 0$ if

$$\forall d \in \mathbb{N} \quad \limsup_{n\to\infty} \frac{|\mathrm{N}^d_{G_n}(X_n)|}{|G_n|} = 0.$$

Something you can safely remove

What is a cluster?

Definition

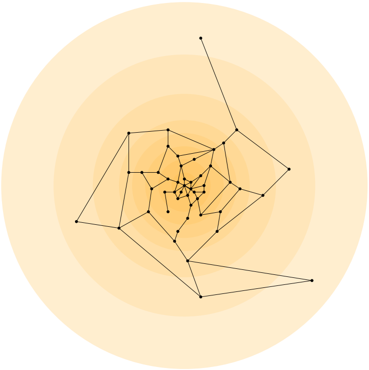
Let G be a local-convergent sequence of graphs.
A sequence X is a *cluster* of G if the following conditions hold:

1. If one marks the elements of X_n in G_n the sequence of marked graphs is still local-convergent;
2. $\partial_{\mathsf{G}}\mathsf{X} \approx 0$ (i.e. the sequence $(\partial_{G_n} X_n)_{n\in\mathbb{N}}$ is negligible).

Remark

- condition 1 means that clusters are not “forced”;
- condition 2 means that clusters can be separated.

Globular Cluster

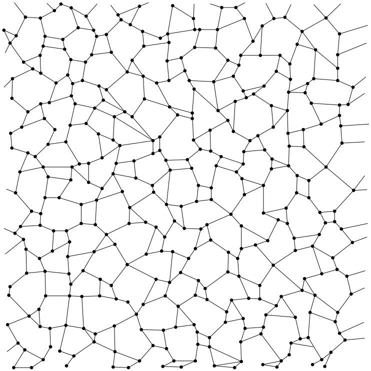


$$\forall \epsilon > 0 \ \exists d \in \mathbb{N} :$$

$$\liminf_{n\to\infty} \sup_{v_n \in X_n} \frac{|\mathrm{N}^d_{G_n}(v_n)|}{|X_n|} > 1-\epsilon.$$

(Almost) connected limit

Residual Cluster



$$\forall d \in \mathbb{N}:$$

$$\limsup_{n\to\infty} \sup_{v_n \in X_n} \frac{|\mathrm{N}^d_{G_n}(v_n)|}{|X_n|} = 0.$$

Zero-measure limit connected components

Marking of all Globular Clusters

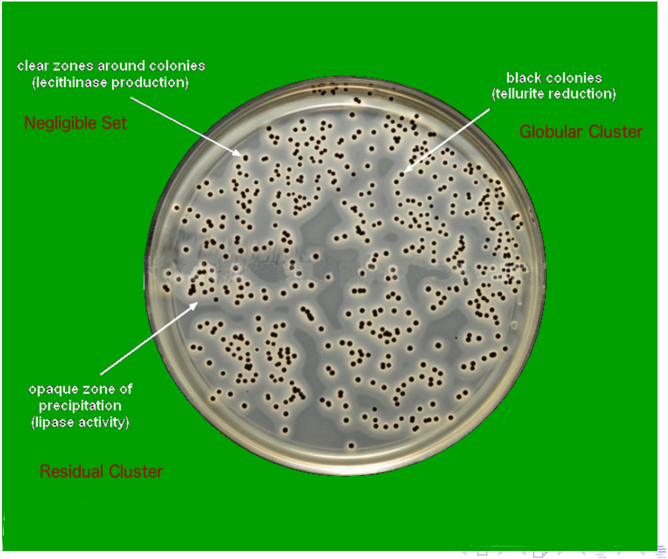
Theorem (Nešetřil, Ossona de Mendez, 2015+)

Let G be a local convergent sequence of graphs. Then there exists (for all n) a marking G_n^+ of G_n by $S, R, M_1, \dots, M_i, \dots$ such that

- marks $S, R, M_1, \dots, M_i, \dots$ induce a partition of $V(G_n)$ and each mark M_i marks one of the connected components of $G_n \setminus S$;
- the sequence G^+ is local convergent;
- $S(\mathsf{G})$ is negligible in G^+;
- $M_i(\mathsf{G})$ is a globular cluster of G^+;
- $R(\mathsf{G})$ is a residual cluster of G^+.

Asymptotic Structure

(Staphylococcus Aureus)



Asymptotic Structure

(Milky Way)

Generic Point

How to transform a random point into a constant?

Theorem (1-point random lift theorem)

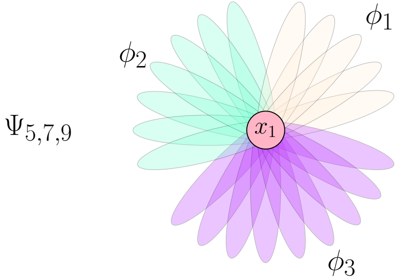
There exists a (unique) continuous function $\widetilde{\Pi} : \mathfrak{M}_\sigma \to \mathrm{P}(\mathfrak{M}_{\sigma^\bullet})$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Rel}(\sigma) & \xrightarrow{\ \Pi\ } & \mathrm{P}(\mathrm{Rel}(\sigma^\bullet)) \\
\Big\downarrow{\scriptstyle \iota^\sigma} & & \Big\downarrow{\scriptstyle \iota_*^{\sigma^\bullet}} \\
\mathfrak{M}_\sigma & \xrightarrow[\ \widetilde{\Pi}\ ]{} & \mathrm{P}(\mathfrak{M}_{\sigma^\bullet})
\end{array}$$

Ingredients of the Proof

Local Stone pairing of ϕ and $\mathbf{A}$ at v:

$$\langle \phi, \mathbf{A} \rangle_v = \Pr(\mathbf{A} \models \phi(v, X_2, \ldots, X_p))$$



$$\langle \Psi_{5,7,9}, \mathbf{A} \rangle = \mathbb{E}_v \Big[\langle \phi_1, \mathbf{A} \rangle_v^5 \, \langle \phi_2, \mathbf{A} \rangle_v^7 \, \langle \phi_3, \mathbf{A} \rangle_v^9 \Big].$$

Characteristic function:

$$\gamma(\mathbf{t}) = \mathbb{E}\big[e^{i\mathbf{t}\cdot\mathbf{D}}\big] = \sum_{w_1 \geq 0} \cdots \sum_{w_d \geq 0} \langle \psi_{\mathbf{w}}, \mathbf{A} \rangle \prod_{j=1}^{d} \frac{(it_j)^{w_j}}{w_j!}.$$

Application: Sizes of the Globular Clusters

Let

$$\varpi_d := \mathrm{dist}(x_1, x_2) \leq d.$$

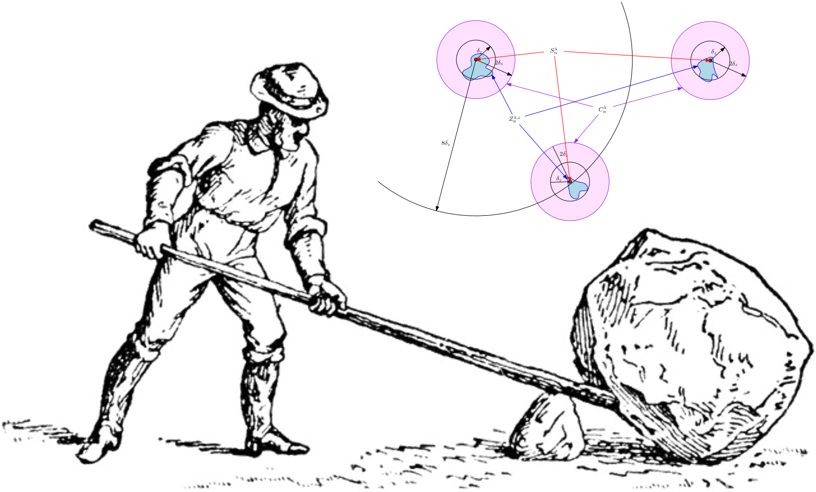
Then

$$m_d(k) = \lim_{n\to\infty} \langle \overbrace{\varpi_d \otimes_{x_1} \cdots \otimes_{x_1} \varpi_d}^{k}, G_n \rangle = \lim_{n\to\infty} \mathbb{E}_v[\langle \varpi, G_n \rangle_v^k].$$

Thus $\forall \lambda > 0$, the number of globular clusters of measure λ is:

$$N(\lambda) = \frac{1}{\lambda} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \left[\sum_{k\geq 1} \lim_{d\to\infty} m_d(k) \frac{(is)^k}{k!} \right] e^{-i\lambda s}\,\mathrm{d}s$$

Keep digging. . .



Details

$\epsilon_z = 2^{-z}$, $z_0(\lambda) = \lceil 5 - 2\log_2 \lambda \rceil$,
$\alpha_1(\lambda) < \alpha_2(\lambda) < \cdots < \lambda < \cdots < \beta_2(\lambda) < \beta_1(\lambda)$ s.t. $\Lambda \cap [\alpha_1(\lambda), \beta_1(\lambda)] = \{\lambda\}$,
$\alpha_z(\lambda), \beta_z(\lambda) \in \mathcal{R}$, $|\beta_z(\lambda) - \alpha_z(\lambda)| < \epsilon_z$.
$\delta_1(\lambda) < \delta_2(\lambda) < \dots$ s.t. $\forall d \geq \delta_z(\lambda)$:

$$\begin{cases} |F_d(\alpha_z(\lambda)) - F(\alpha_z(\lambda))| < \epsilon_z \\ |F_d(\beta_z(\lambda)) - F(\beta_z(\lambda))| < \epsilon_z \end{cases}$$

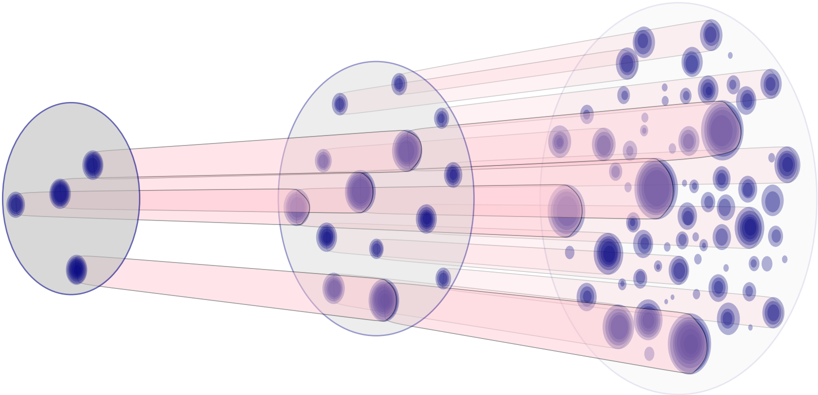
$\eta_1(\lambda) < \eta_2(\lambda) < \dots$ s.t. $\forall z \in \mathbb{N}$, $\forall n \geq \eta_z(\lambda)$ and $\forall k \in \{1, \dots 8\}$:

$$\begin{cases} |F_{k\delta_z(\lambda),n}(\alpha_z(\lambda)) - F_{k\delta_z(\lambda)}(\alpha_z(\lambda))| < \epsilon_z \\ |F_{k\delta_z(\lambda),n}(\beta_z(\lambda)) - F_{k\delta_z(\lambda)}(\beta_z(\lambda))| < \epsilon_z. \end{cases}$$

$Z_n^{\lambda,z} = \Big\{v : D_{8\delta_z,n}(v) \leq \beta_z(\lambda) \text{ and } D_{\delta_{z'},n}(v) > \alpha_{z'}(\lambda)\ (\forall z' \in \{z_0(\lambda), \dots, z\})\Big\}$.
S_n^λ =maximal set of vertices $v \in Z_n^{\lambda,z}$, pairwise at distance at least $7\delta_z$, where $\eta_z \leq n < \eta_{z+1}$.
and eventually…

$$C_n^\lambda = \begin{cases} \emptyset, & \text{if } n < \eta_{z_0(\lambda)} \\ \mathrm{N}_{\mathbf{G}_n}^{2\delta_z}(S_n^\lambda), & \text{otherwise, if } z \text{ is such that } \eta_z \leq n < \eta_{z+1} \end{cases}$$

Cluster Structure



Typical shape of a structure sequence continuously segmented by a clustering.

Modelings

Modelings

Definition

A *modeling* $\mathbf{A}$ is a graph on a standard probability space s.t. every first-order definable set is measurable.

The Stone pairing extends to modelings:

$$\langle \phi, \mathbf{A} \rangle = \nu_{\mathbf{A}}^{\otimes p}(\phi(\mathbf{A})).$$

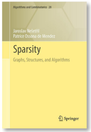
By Fubini's theorem, it holds:

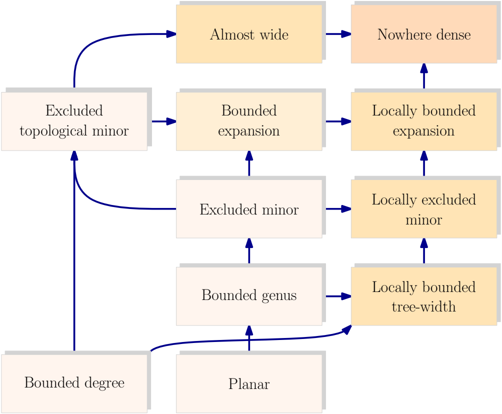
$$\langle \phi, \mathbf{A} \rangle = \int \cdots \int \mathbf{1}_{\phi(\mathbf{A})}(x_1, \dots, x_p)\, \mathrm{d}\nu_{\mathbf{A}}(x_1) \dots \mathrm{d}\nu_{\mathbf{A}}(x_p).$$

Modelings as FO-limits?

Theorem (Nešetřil, Ossona de Mendez 2013)

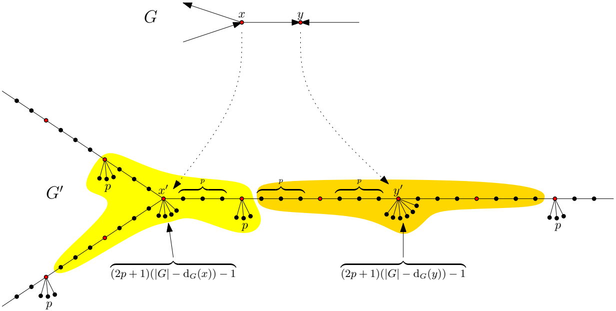
If a monotone class $\mathcal{C}$ has modeling FO-limits then the class $\mathcal{C}$ is nowhere dense.





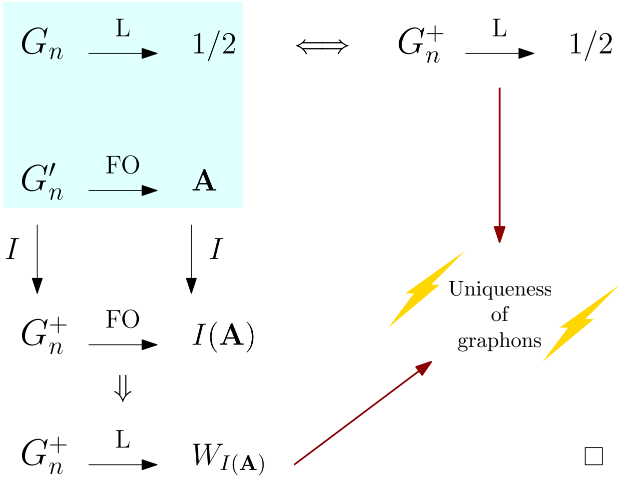
Proof (sketch)

- Assume $\mathcal{C}$ is somewhere dense. There exists $p \geq 1$ such that $\mathrm{Sub}_p(K_n) \in \mathcal{C}$ for all n;
- For an oriented graph G, define $G' \in \mathcal{C}$:



- $\exists$ basic interpretation I, such that for every graph G, $I(G') \cong G[k(G)] \overset{\mathrm{def}}{=} G^+$, where $k(G) = (2p+1)|G|$.

Proof (sketch)



$\square$

Modelings as FO-limits?

Theorem (Nešetřil, Ossona de Mendez 2013)

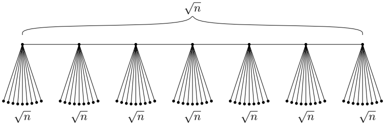
If a monotone class $\mathcal{C}$ has modeling FO-limits then the class $\mathcal{C}$ is nowhere dense.

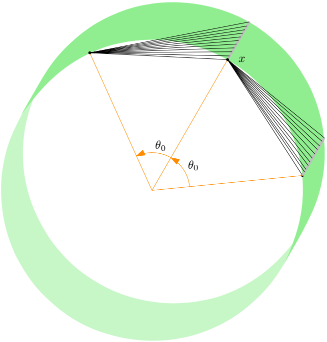
Conjecture (Nešetřil, Ossona de Mendez)

Every nowhere dense class has modeling FO-limits.

- true for bounded degree graphs (Nešetřil, Ossona de Mendez 2012)
- true for bounded tree-depth graphs (Nešetřil, Ossona de Mendez 2013)
- true for trees (Nešetřil, Ossona de Mendez 2016)
- true for plane trees and for graphs with bounded pathwidth (Gajarský, Hliněný, Kaiser, Kráľ, Kupec, Obdržálek, Ordyniak, Tůma 2016)

Example I


$\sqrt{n}$
$\sqrt{n}$
$\sqrt{n}$
$\sqrt{n}$
$\sqrt{n}$
$\sqrt{n}$
$\sqrt{n}$
$\sqrt{n}$


x
θ_0
θ_0

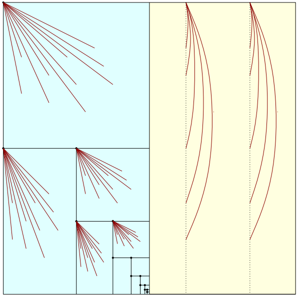
Example II

$$G_n = \overbrace{S_{2^{2^n}(2^{-1}+2^{-n})} + \cdots + S_{2^{2^n}(2^{-i}+2^{-n})} + \cdots + S_{2^{2^n}(2^{-2^n}+2^{-n})}}^{2^n \text{ stars}}$$

Example II

$$G_n = \overbrace{S_{2^{2^n}(2^{-1}+2^{-n})} + \cdots + S_{2^{2^n}(2^{-i}+2^{-n})} + \cdots + S_{2^{2^n}(2^{-2^n}+2^{-n})}}^{2^n \text{ stars}}$$

Big components



Small components

Friedman's $\mathcal{L}(\mathrm{Q_m})$ Logic

First-Order Logic + special quantifier $\mathrm{Q_m}$ with intended interpretation

$$\mathbf{M} \models \mathrm{Q_m} x\ \psi(x, \overline{a}) \iff \{x \in M :\ \mathbf{M} \models \psi(x, \overline{a})\} \text{ is not of measure } 0.$$

System of rules of inference $\mathrm{K_m}$

Theorem (Friedman '79, Steinhorn '85)

A set of sentences T in $\mathcal{L}(\mathrm{Q_m})$ has a totally Borel model if and only if T is consistent in $\mathrm{K_m}$.

Modeling FO_1-Limits

Theorem (Nešetřil, POM 2016+)

Every FO_1-convergent sequence $(G_n)_{n\in\mathbb{N}}$ of graphs (or structures with countable signature) has a modeling FO_1-limit $\mathbf{L}$.
If $(G_n)_{n\in\mathbb{N}}$ is FO-convergent then $\forall\phi$ it also holds

$$\langle\phi,\mathbf{L}\rangle = 0 \quad\iff\quad \lim_{n\to\infty}\langle\phi,G_n\rangle = 0.$$

We denote this by

$$G_n \xrightarrow{\mathrm{FO}_1^*} \mathbf{L}.$$

Sketch of the Proof

- Construct a limit **U** as an ultraproduct with a Loeb measure;
- The structure **U** is a model of the $\mathbf{L}(\mathrm{Q_m})$-theory, which is the union of the complete FO theory and sentences

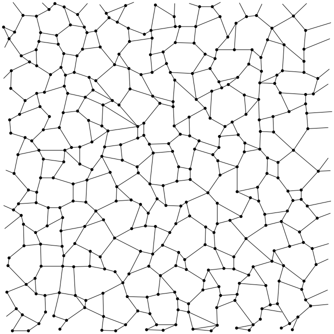
$$\mathrm{Q_m} x_1 \ldots \mathrm{Q_m} x_p \ \phi(x_1, \ldots, x_p)$$

  for each ϕ such that $\lim_{n\to\infty}\langle\phi, G_n\rangle > 0$.
- Let **L** be a totally Borel model.
- For $r \in \mathbb{N}$ let $\theta_1^r, \ldots, \theta_{N(r)}^r$ be the 1-types of rank r. Define

$$\pi_r(X) = \sum_{i \in \lambda(\theta_i^r(\mathbf{L})) \neq 0} \frac{\lambda(X \cap \theta_i^r(\mathbf{L}))}{\lambda(\theta_i^r(\mathbf{L}))} \lim_{n\to\infty} \langle \theta_i^r, G_n \rangle.$$

- The desired probability measure is weak limit π of π_r.

Modeling Limits of Residual Sequences



$$\forall d \in \mathbb{N}:$$

$$\lim_{n\to\infty} \sup_{v_n \in G_n} \frac{|\mathrm{N}^d_{G_n}(v_n)|}{|G_n|} = 0.$$

Zero-measure limit connected components

Theorem (Nešetřil, POM 2016+)

Every residual FO-convergent sequence $(G_n)_{n\in\mathbb{N}}$ of graphs has a modeling FO-limit $\mathbf{L}$.

Modeling Limits of Quasi-Residual Sequences

(G_n) is (d,ϵ)-residual if

$$\lim_{n\to\infty}\sup_{v_n\in G_n}\frac{|\mathrm{N}^d_{G_n}(v_n)|}{|G_n|}<\epsilon.$$

(G_n) is quasi-residual if $\forall d,\epsilon>0\ \exists(S_n)$ s.t. $|S_n|\leq N(d,\epsilon)$ and (G_n-S_n) is (d,ϵ)-residual.

(G_n) is marked quasi-residual if $S_n=\{c_1,\dots,c_{N(d,\epsilon)}\}$ and marks Z_d s.t. $Z_d(G_n)=\{c_1,\dots,c_{F(d,n)}\}$ with

$$\lim_{n\to\infty}\frac{\big|B_d(G_n,\{c_1,\dots,c_{F(d,n)}\})\big|}{|G_n|}=\lim_{m\to\infty}\lim_{n\to\infty}\frac{\big|B_d(G_n,\{c_1,\dots,c_m\})\big|}{|G_n|}.$$

Modeling Limits of Quasi-Residual Sequences

Lemma

If

- (G_n) is marked quasi-residual $(4d, \epsilon)$-residual
- $G_n \xrightarrow{\mathrm{FO}_1^*} \mathbf{L}$

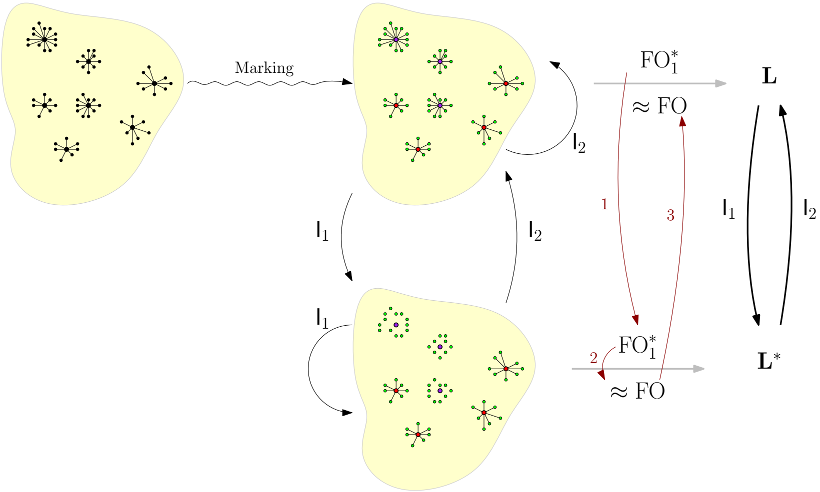
then $\mathbf{L}$ is (d, ϵ)-residual.

Lemma

Assume $(G_n)_{n\in\mathbb{N}}$ is FO-convergent and G_n is $(2d, \epsilon)$-residual. If $G_n \xrightarrow{\mathrm{FO}_1} \mathbf{L}$ and $\mathbf{L}$ is $(2d, \epsilon)$-residual then $\forall d$-local formula ϕ with p free variables it holds

$$|\langle \phi, \mathbf{L}\rangle - \lim_{n\to\infty} \langle \phi, G_n\rangle| < p^2\epsilon.$$

Modeling Limits of Quasi-Residual Sequences



Modeling Limits of Nowhere Dense Sequences

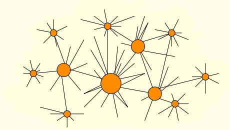
Theorem (Nešetřil, POM 2016+)

Every FO-convergent quasi-residual sequence of graphs has a modeling FO-limit.

Theorem (Nešetřil, POM 2016)

A hereditary class of graphs $\mathcal{C}$ is nowhere dense if and only if $\forall d, \forall \epsilon > 0, \forall G \in \mathcal{C}, \exists S \subseteq G$ with $|S| \leq N(d, \epsilon)$ such that

$$\sup_{v \in G - S} \frac{|B_d(G - S, v)|}{|G|} \leq \epsilon.$$



Theorem (Nešetřil, POM 2016+)

A monotone class $\mathcal{C}$ is nowhere dense if and only if every FO-convergent sequence of graphs in $\mathcal{C}$ has a modeling FO-limit.

Perspectives

Local-Global Convergence

- Defined from colored neighborhood metric (Bollobás and Riordan '11)

Definition (General Setting)

Let σ, σ^+ be countable signature with $\sigma \subseteq \sigma^+$, and let X be a fragment of $\mathrm{FO}(\sigma^+)$.
A sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is *X-local global convergent* if the sequence of the sets

$$\Omega_{\mathbf{A}_n} = \{\mathbf{A}_n^+ : \mathrm{Shadow}(\mathbf{A}_n^+) = \mathbf{A}_n\}$$

converges with respect to Hausdorff distance.

Properties

- (Using Blaschke theorem):

Every sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ has an X-local global convergent subsequence.

- FO_0-local-global convergence. (Using Fagin theorem):

For every NP property π,

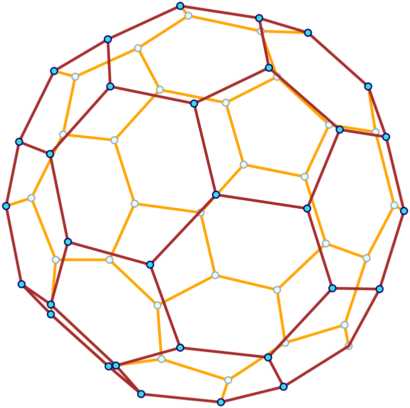
  - either all but finitely many G_n satisfy π;
  - or all but finitely many G_n do not satisfy π.

- $\mathrm{FO}^{\mathrm{local}}$-local-global convergence with monadic lifts.

This is standard local-global convergence.

→ graphings are still limits of graphs with bounded degrees (Hatami, Lovász, and Szegedy '14)
→ allows a finer study of the residue and marking of expander parts.

Expanding Cluster



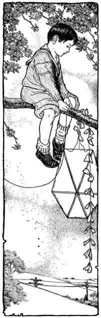
$$\forall \epsilon > 0\ \exists d \in \mathbb{N}:$$

$$\forall \mathsf{Z} \subseteq \mathsf{X} \text{ with } |Z_n| > \epsilon |X_n|$$

$$\liminf_{n\to\infty} \frac{|\mathrm{N}^d_{\mathbf{A}_n}(Z_n)|}{|X_n|} > 1 - \epsilon.$$

For bounded degree:
$\iff \quad \forall \epsilon > 0 \quad \exists \mathsf{N}_\epsilon \subseteq \mathsf{X}$, such that

- $|\mathsf{N}_\epsilon| < \epsilon |\mathsf{C}|$;
- $\mathbf{G}[\mathsf{X} \setminus \mathsf{N}_\epsilon]$ is a vertex expander sequence.



Thank you for your attention.