Workshop on Algorithmic Graph Structure Theory

University of Warsaw, July 14 2017

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INTERNATIONAL COLLOQUIUM
ON AUTOMATA, LANGUAGES
AND PROGRAMMING

University of Warsaw, July 10-13 2017

On Neighbourhood Complexity and Kernels for Distance-r Dominating Sets on Nowhere Dense Classes of Graphs

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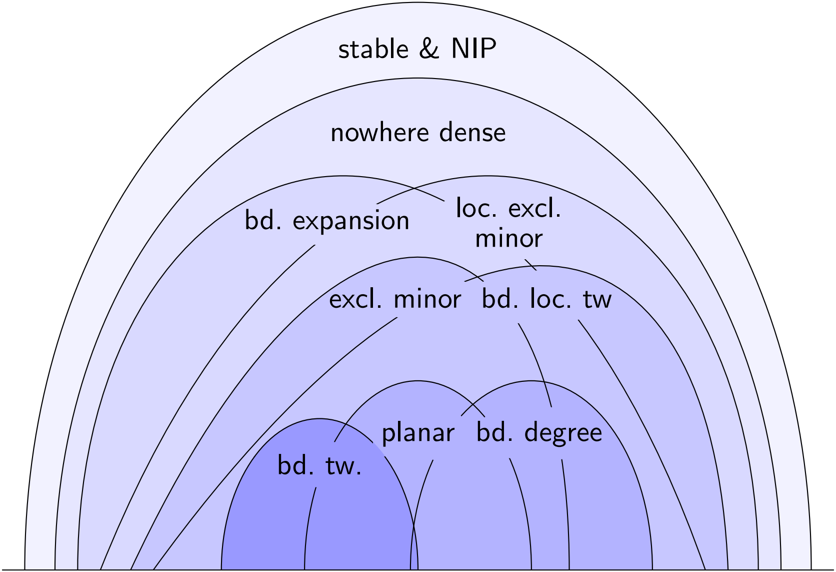
Graph Classes

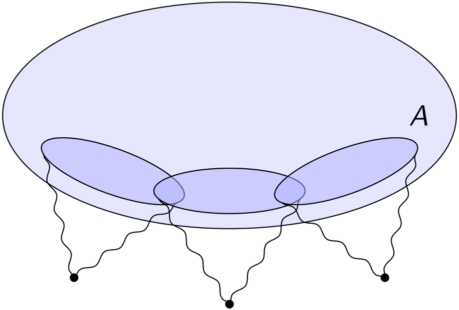


Neighbourhood Complexity

Definition

Let G be a graph and let $r \in \mathbb{N}$. The *r-neighbourhood complexity* of a subset $A \subseteq V(G)$ in G is

$$\nu_r(G, A) = |\{N_r(v) \cap A : v \in V(G)\}|.$$



Theorem (Sauer-Shelah Lemma) [VC71, S72, S72]

If G has VC-dimension d, then

$$\nu_1(G, A) \leq |A|^d.$$

Theorem [Gajarski et al. 2013]

If $\mathcal{C}$ is nowhere dense and $G \in \mathcal{C}$, then

$$\nu_1(G, A) \leq f(\varepsilon) \cdot |A|^{1+\varepsilon}.$$

Theorem [RSS 2016]

If $\mathcal{C}$ has bounded expansion and $G \in \mathcal{C}$, then

$$\nu_r(G, A) \leq f(r) \cdot |A|.$$

This talk

Theorem [EGKKPRS 16+]

A class $\mathcal{C}$ is nowhere dense if and only if there is $f : \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ such that for all $r \in \mathbb{N}$, $\varepsilon > 0$, $G' \subseteq G \in \mathcal{C}$ and $A \subseteq V(G)$

$$\nu_r(G', A) \le f(r, \varepsilon) \cdot |A|^{1+\varepsilon}.$$

An application [EGKKPRS 16+]

If $\mathcal{C}$ is nowhere dense then there is a function $f : \mathbb{N} \to \mathbb{R}$ and a polynomial time algorithm which on input $G \in \mathcal{C}$, $k, r \in \mathbb{N}$, $\varepsilon > 0$, either concludes that $\gamma_r(G) > k$, or computes a subgraph G' and $Z \subseteq V(G')$ such that

- $|V(G')| \le g(r, \varepsilon) \cdot k^{1+\varepsilon}$ and
- G has a distance-r dominating set of size k if and only if Z has a distance-r dominating set in G' of size k.

The Distance-r Dominating Set Problem

Linear kernels on

- Planar (dominating set) [Albers et al. 04]
- Bounded genus (dominating set) [Bodlaender et al. 09]
- Apex-minor free (dominating set) [Fomin et al. 10]
- H-minor free (dominating set) [Fomin et al. 12]
- H-topological minor free (dominating set) [Fomin et al. 13]
- Bounded expansion (distance-r dominating set) [Drange et al. 14]

Almost linear kernels on nowhere dense classes for distance-r dominating set for every value of r.

Proof Outline

Theorem [EGKKPRS 16+]

A class $\mathcal{C}$ is nowhere dense if and only if there is $f : \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ such that for all $r \in \mathbb{N}$, $\varepsilon > 0$, $G' \subseteq G \in \mathcal{C}$ and $A \subseteq V(G)$

$$\nu_r(G', A) \leq f(r, \varepsilon) \cdot |A|^{1+\varepsilon}.$$

1. Reduce the graph size to $|A|^k$, for a constant k, using NIP.
2. Identify interfaces of size $|A|^{\varepsilon/2}$ which control all connections to A using weak colouring numbers.
3. Show that there are at most $|A|^{1+\varepsilon/2}$ interfaces using uniform quasi-wideness and weak colouring numbers.

Independence and Number of Types

Definition

Let G be a graph, $A \subseteq V(G)$ and $r \in \mathbb{N}$. The *distance-r-profile of $v \in V(G) \setminus A$ on A* is $(d_1, \dots, d_{|A|}) \in \{1, \dots, r, \infty\}^{|A|}$, where

$$d_i = \begin{cases} \mathrm{dist}(v, a_i) & \text{if } \mathrm{dist}(v, a_i) \leq r \\ \infty & \text{otherwise.} \end{cases}$$

Theorem

If $\mathcal{C}$ is NIP, then for all $r \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that for all $G \in \mathcal{C}$ and $A \subseteq V(G)$, the number of realised distance-r-profiles is bounded by $|A|^k$.

Reducing Graph Size

Corollary

If $\mathcal{C}$ is NIP, then $\nu_r(G, A) \leq |A|^k$ for some k depending on r only.

Corollary

If $\mathcal{C}$ is NIP, then on input $G \in \mathcal{C}$, $A \subseteq V(G)$ and $r \in \mathbb{N}$, we can compute in polynomial time a subgraph $G' \subseteq G$ with $A \subseteq V(G')$ and $|V(G')| \leq r \cdot |A|^{k+1}$, for some k, such that $\nu_r(G', A) \geq \nu_r(G, A)$.

- Take one vertex v_κ from each equivalence class κ and for each $a \in A \cap N_r(v_\kappa)$ a path of length at most r from v_κ to a.

Proof Outline

Theorem [EGKKPRS 16+]

A class $\mathcal{C}$ is nowhere dense if and only if there is $f : \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ such that for all $r \in \mathbb{N}$, $\varepsilon > 0$, $G' \subseteq G \in \mathcal{C}$ and $A \subseteq V(G)$

$$\nu_r(G', A) \leq f(r, \varepsilon) \cdot |A|^{1+\varepsilon}.$$

1. Reduce the graph size to $|A|^k$, for a constant k, using NIP. ✓
2. Identify interfaces of size $|A|^{\varepsilon/2}$ which control all connections to A using weak colouring numbers.
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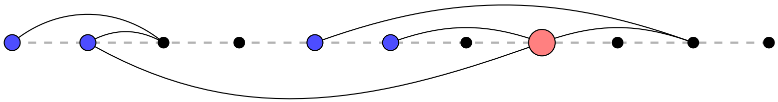
Generalised Colouring Numbers

Is there an order L of $V(G)$ such that every vertex $v \in V(G)$ can reach only few smaller vertices w by paths of length at most r such that w is the smallest vertex on the path?



Definition

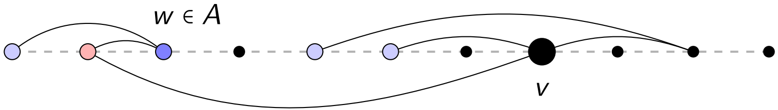
For $v \in V(G)$, let $\mathrm{WReach}_r[G, L, v]$ be the set of vertices reachable from v in the above way. Define

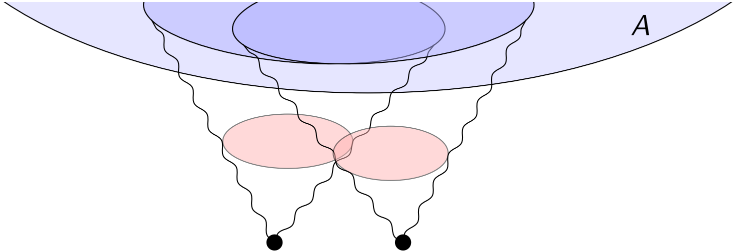
$$\mathrm{wcol}_r(G) := \min_{L \text{ L.O.}} \max_{v \in V(G)} |\mathrm{WReach}_r[G, L, v]|.$$

Generalised Colouring Numbers

Theorem

Let G be a graph, $A \subseteq V(G)$ and $v \in V(G)$. Then $\mathrm{WReach}_r[G, L, v] \cap \mathrm{WReach}_r[G, L, A]$ intersects every path of length at most r between v and A.





Generalised Colouring Numbers

Theorem

If $\mathcal{C}$ is nowhere dense, then there is $f : \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ such that for all $G' \subseteq G \in \mathcal{C}$, all $r \in \mathbb{N}$ and $\varepsilon > 0$ we have

$$\mathrm{wcol}_r(G') \leq f(r, \varepsilon) \cdot |V(G')|^{\varepsilon}.$$

Corollary

The graph G' we constructed satisfies $\mathrm{wcol}_{2r}(G') \leq f(2r, \varepsilon/(2k)) \cdot |A|^{\varepsilon/2}$.

Proof Outline

Theorem [EGKKPRS 16+]

A class $\mathcal{C}$ is nowhere dense if and only if there is $f : \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ such that for all $r \in \mathbb{N}$, $\varepsilon > 0$, $G' \subseteq G \in \mathcal{C}$ and $A \subseteq V(G)$

$$\nu_r(G', A) \leq f(r, \varepsilon) \cdot |A|^{1+\varepsilon}.$$

1. Reduce the graph size to $|A|^k$, for a constant k, using NIP. ✓
2. Identify interfaces of size $|A|^{\varepsilon/2}$ which control all connections to A using weak colouring numbers. ✓
3. Show that there are at most $|A|^{1+\varepsilon/2}$ interfaces using uniform quasi-wideness and weak colouring numbers.

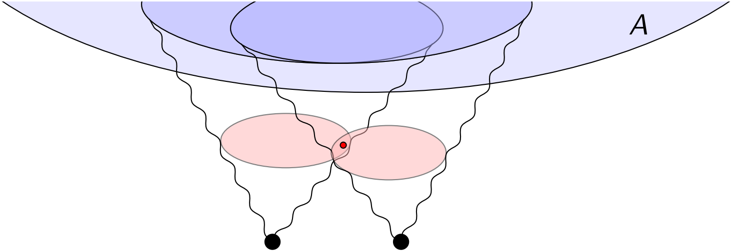
Bounding the Number of Interfaces

Theorem

Let L be an order such that $\mathrm{WReach}_{2r}[G, L, v]$ is small. Denote by $\mathcal{Y}$ the family of sets of the form $Y[v] = \mathrm{WReach}_r[G, L, v] \cap \mathrm{WReach}_r[G, L, A]$. Define

$$\gamma : \mathcal{Y} \to \mathrm{WReach}_r[G, L, A] : Y[v] \mapsto \max Y.$$

Then $\gamma^{-1}(w) \subseteq \mathrm{WReach}_{2r}[G, L, w]$ for all $w \in \mathrm{WReach}_r[G, L, A]$.



Bounding the Number of Interfaces

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Then $\gamma^{-1}(w) \subseteq \mathrm{WReach}_{2r}[G, L, w]$ for all $w \in \mathrm{WReach}_r[G, L, A]$.

Corollary

There are at most $2^{\mathrm{wcol}_{2r}(G)} \cdot f(r, \varepsilon) \cdot |A|^{1+\varepsilon}$ many interfaces.

Improving the Bound

Theorem

Let $\mathcal{C}$ be nowhere dense and let $r \in \mathbb{N}$. There is a number d such that for all $G \in \mathcal{C}$ and all orders L of $V(G)$ it holds that the graph G' with $V(G') = V(G)$ and $\{u, v\} \in E(G')$ if and only if $u \in \mathrm{WReach}_r[G, L, v]$ has VC-dimension at most d.

Corollary

Let $G \in \mathcal{C}$ and $Y \subseteq V(G)$. Then there are at most $|Y|^d$ subsets of A of the form $Y[v] = Y \cap \mathrm{WReach}_r[G, L, v]$ for $v \in V(G)$.

Corollary

By rescaling ε again we get the main theorem.

Uniform Quasi-Wideness

Definition

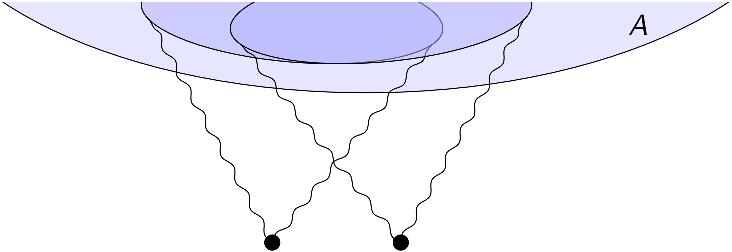
A class $\mathcal{C}$ of graphs is *uniformly quasi-wide* if there are functions $N : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $s : \mathbb{N} \to \mathbb{N}$ such that for all $r, m \in \mathbb{N}$ and all subsets $A \subseteq V(G)$ for $G \in \mathcal{C}$ of size $|A| \geq N(r, m)$ there is a set $S \subseteq V(G)$ of size $|S| \leq s(r)$ and a set $B \subseteq A$ of size $|B| \geq m$ which is r-independent in $G - S$.

Theorem

A class $\mathcal{C}$ of graphs is uniformly quasi-wide if and only if it is nowhere dense.

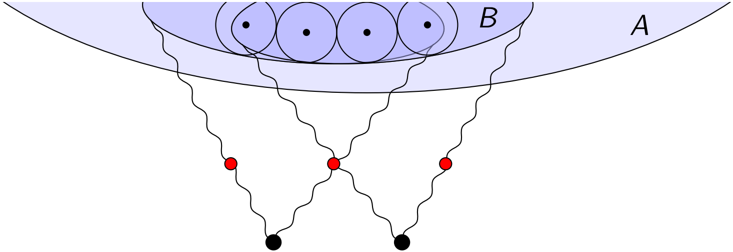
The Final Proof

- Assume the VC-dimension of G' is high. Take a large subset A which is shattered in G', that is, for each subset $X \subseteq A$ we have a vertex $v \in V(G)$ with $X = \mathrm{WReach}_r[G, L, v] \cap A$.

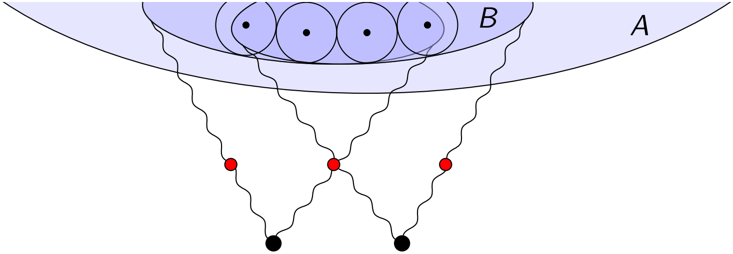


The final proof

- As G comes from a nowhere dense class, there is a large subset $B \subseteq A$ which is $2r$-independent after removing a set S of size at most $s(2r)$ from G.



The final proof



- The set B, as a subset of A, must also be shattered, however, every vertex v weakly reaching two elements of B must make its connection via S (otherwise B is not $2r$-independent).
- There are not enough different ways to connect to the constant size set S to shatter the large set B. A contradiction.

Summary

Theorem [EGKKPRS 16+]

A class $\mathcal{C}$ is nowhere dense if and only if there is $f : \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ such that for all $r \in \mathbb{N}$, $\varepsilon > 0$, $G' \subseteq G \in \mathcal{C}$ and $A \subseteq V(G)$

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