Stability in Graphs

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Algorithmic Meta-Theorems

Rephrased in parameterized complexity. Let $\mathcal{C}$ be a class of graphs.

Then the following problem is fixed parameter tractable

MC($\mathcal{L}$, $\mathcal{C}$)

Input: Graph $G \in \mathcal{C}$, formula $\varphi \in \mathcal{L}$

Parameter: $|\varphi|$ *(or* $|\varphi| + tw(G)$ *or* $|\varphi| + excluded\ K_t$ *)*

Problem: Decide $G \vDash \varphi$?

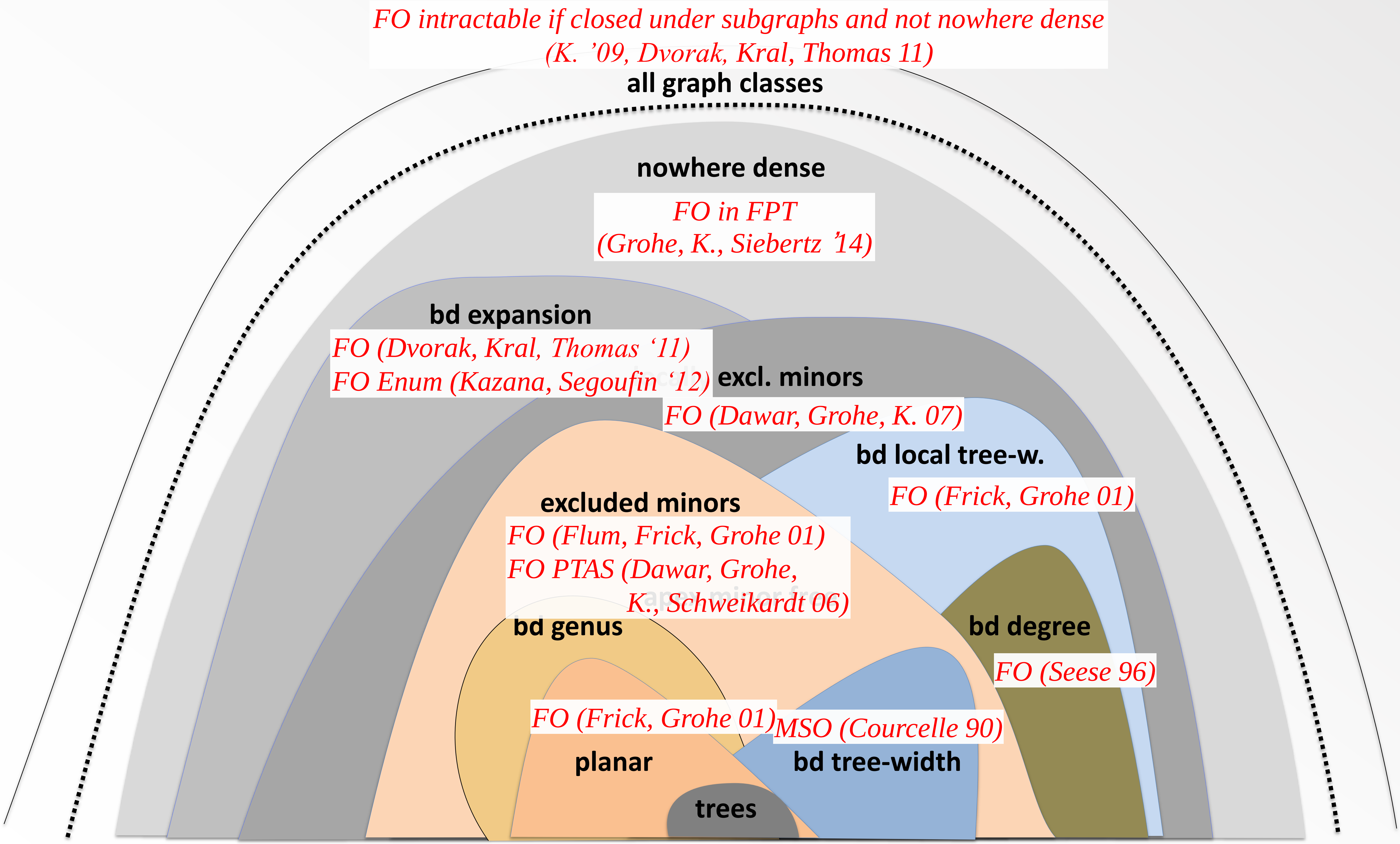
Research programme.

For important logics $\mathcal{L}$ such as first-order or monadic second-order logic:

identify structural parameter $\mathcal{P}$ such that

MC($\mathcal{C}$, $\mathcal{L}$) is FPT for a class $\mathcal{C}$ if, and only if, $\mathcal{C}$ has property $\mathcal{P}$.

Algorithmic Meta-Theorems



Nowhere Dense Classes of Graphs

Definition. (Nešetril, Ossona de Mendez)

A class $\mathcal{C}$ of graphs is *nowhere dense* if for every $r \geq 1$ there is a number $f(r)$ such that $K_{f(r)} \not\preccurlyeq_r G$ for all $G \in \mathcal{C}$.

If the function $f : r \to f(r)$ is computable then we call $\mathcal{C}$ *effectively* nowhere dense.

Examples.

- Graph classes excluding a fixed minor
- Graph classes of bounded local tree width or locally excluding a minor.
- Classes of bounded expansion.

Non-Examples.

- 2-degenerate graphs.
- Interval graphs
- Partial orders
- Classes of bounded rank or clique width.

Model Checking on Nowhere Dense Classes

Theorem.* *(Grohe, K., Siebertz ‘14)

Every problem definable in first-order logic can be decided in time $O(n^{1+\epsilon})$, for every $\varepsilon > 0$, on any class of graphs that is nowhere dense.

Theorem. ***(K. 09, Dvorak, Kral, Thomas ’11)***

If a class C closed under subgraphs is not nowhere dense, then FO-model-checking is not fixed-parameter tractable (unless AW[$*$] = FPT).

Corollary.

Let $\mathcal{C}$ be a class of graphs closed under taking subgraphs.

$MC(FO, \mathcal{C}) \in FPT$ if, and only if, $\mathcal{C}$ is nowhere dense.

Future Work

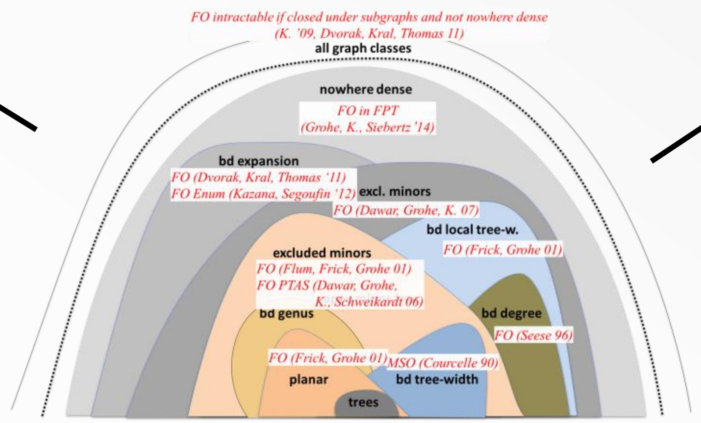
What if our classes are no longer sparse? And no longer closed under taking subgraphs?

Wait till tomorrow.

Extension into the Dense World

Option 1.
Do, what we always do

Option 2:
Interpret, what we have done



Option 3.
Make it stable

Option 1: Do, what we do

Option 1.

Look at know and interesting classes of dense graphs and try to prove efficient FO-model checking there.

Theorem. (Ganian, Hlineny, Kral, Obdrzalek, Schwartz, Teska '13)

For every finite subset L of reals and every FO sentence Φ, there exists an algorithm running in time $O(n \log n)$ that decides whether an input n-vertex L-interval graph G given by its L-representation satisfies Φ.

Theorem. (Gajarsky, Hlineny, Lokshtanov, Obdrzalek, Ordyniak, Ramanujan, Saurabh '15)

Let $P = (P, \leq)$ be a poset of width w, with elements coloured by $\lambda : P \to \Lambda$ where Λ is a finite set, and let φ be an FO sentence in negation normal form.

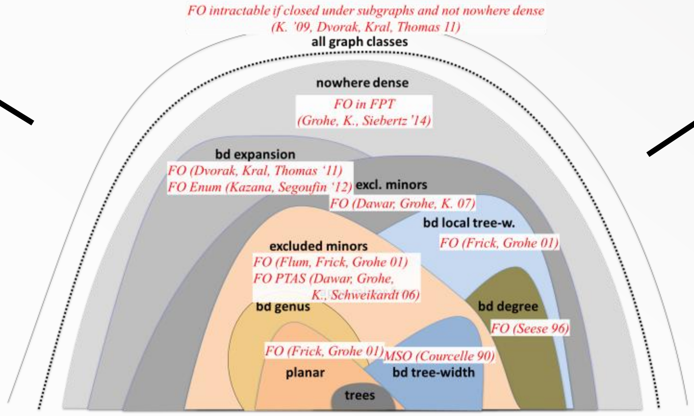
There is an algorithm which decides whether $P \models \varphi$ in time $f(w, \varphi) \cdot ||P||^2$.

Problem. *There is no clear candidate for an optimal class with tractable MC(FO).*

Extension into the Dense World

Option 1.
Do, what we always do

Option 2:
Interpret, what we have done



Option 3.
Make it stable

Interpretations

Observation.

Let $\mathcal{C}$ be a class of graphs such that $MC(\mathcal{C}, FO) \in FPT$.

For $G \in \mathcal{C}$ and define $\bar{G} := \left(V(G), \overline{E(G)}\right)$.

Here: $\overline{E(G)}$ is the complement set, i.e. flip edges and non-edges.

Let $\bar{\mathcal{C}} := \{\, \bar{G} : G \in \mathcal{C}\}$.

Claim. $MC(FO, \bar{\mathcal{C}}) \in FPT$.

Given $\bar{G} \in \bar{\mathcal{C}}$ and a formula $\varphi \in FO$:

Replace in φ every $E(x, y)$ by $\neg E(x, y)$ to obtain formula $\varphi' \in FO$.

Then $\bar{G} \models \varphi$ if, and only if, $G \models \varphi'$.

$G \models \varphi'$ can be decided efficiently.

Note that $\bar{\mathcal{C}}$ may not be sparse, even if $\mathcal{C}$ is.

Interpretations

Definition.

A (simple) first-order interpretation Θ in graphs consists of a formula $\theta(x, y)$.

Given a graph G, Θ defines a new graph $\Theta(G) \coloneqq (V(G), \{\, \{u, v\} : G \vDash \theta[u, v]\})$.

Example.

Take Θ defined by $\theta(x, y) \coloneqq \neg E(x, y)$. Then $\Theta(G) = \bar{G}$.

Note. General interpretations can add new vertices and, e.g., define subdivisions.

Interpretation lemma.

Given a formula $\varphi \in FO$ and a graph G:

Replace in φ every $E(x, y)$ by $\theta(x, y)$ to obtain formula $\Theta(\varphi) \in FO$.

Then for every graph G: $\Theta(G) \vDash \varphi$ if, and only if, $G \vDash \Theta(\varphi)$.

Interpretations

Interpretation lemma.

Given a formula $\varphi \in FO$ and a graph G:

Replace in φ every $E(x,y)$ by $\theta(x,y)$ to obtain formula $\Theta(\varphi) \in FO$.

Then for every graph G: $\Theta(G) \models \varphi$ if, and only if, $G \models \Theta(\varphi)$.

Model checking idea:

Let $\mathcal{C}$ be a class of graphs such that $MC(\mathcal{C}, FO) \in FPT$.

Let $\Theta(\mathcal{C}) := \{\Theta(G) : G \in \mathcal{C}\}$.

Then $MC(\Theta(\mathcal{C}), FO)$ should be fixed-parameter tractable.

Given $\Theta(G) \in \Theta(\mathcal{C})$ and a formula $\varphi \in FO$:

1. Compute $\Theta(\varphi)$.
2. Then $\Theta(G) \models \varphi$ if, and only if, $G \models \Theta(\varphi)$.
3. $G \models \Theta(\varphi)$ can be decided efficiently.

Problem. How do we compute G from $\Theta(G)$?

Interpretations in Bounded Degree Graphs

Definition. (Gajarský, Hlineňý, Obdržálek, Lokshtanov, Ramanujan '16)

For a graph G and $k \in \mathbb{N}$, the *near-k-twin relation* of G is the relation ρ_k on $V(G)$ defined by $(u, v) \in \rho_k \Leftrightarrow |N(u) \triangle N(v)| \leq k$.

Definition.

1. A graph G is (k_0, p)-*near-uniform* if there exists $k \leq k_0$ for which the near-k-twin relation of H is an equivalence relation of index at most p.
2. A graph class $\mathcal{C}$ is (k_0, p)-*near-uniform* if every member of $\mathcal{C}$ is (k_0, p)-near-uniform, and $\mathcal{C}$ is *near-uniform* if there exist integers k_0 *and* p such that $\mathcal{C}$ is (k_0, p)-near-uniform.

Theorem. (Gajarský, Hlineňý, Obdržálek, Lokshtanov, Ramanujan '16)

Let $\mathcal{C}$ be a (k_0, p)-near-uniform graph class for some $k_0, p \in \mathbb{N}$.
Then the FO model checking problem on $\mathcal{C}$ is fixed- parameter tractable.

Theorem. (k_0, p)-near-uniform graph class are exactly those interpretable in classes of bounded degree.

Interpretations in Sparse Classes

Theorem.

(k_0, p)-near-uniform graph class are exactly those interpretable in classes of bounded degree.

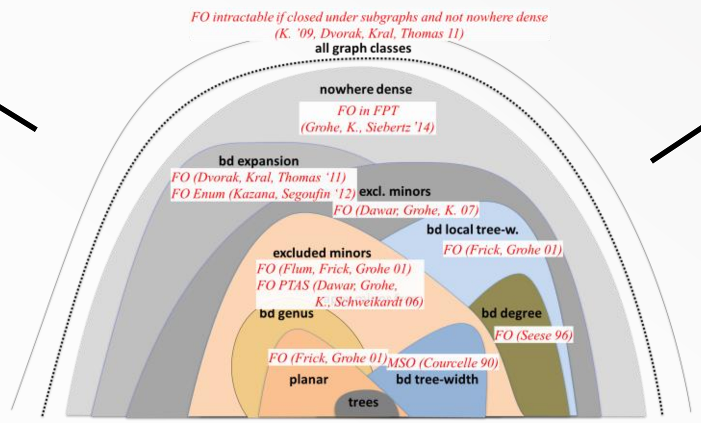
Question.

- Can we characterise classes interpretable in other sparse classes?
- What about classes interpretable in nowhere dense classes of graphs?

Extension into the Dense World

Option 1.
Do, what we always do

Option 2:
Interpret, what we have done



Option 3.
Make it stable

Characterisations of Nowhere Dense Graphs

We know that nowhere density has many equivalent characterisations.

Equivalent definitions of nowhere dense.

- Can we find a characterisation that has a meaningful definition on dense graphs also?
- In fact: tractability of FO on nowhere dense classes could merely be an artefact of tractability of FO on a much larger class of graph classes happens to coincide with nowhere dense classes if closed under subgraphs.

Stability and the NIP

Theorem. (Adler, Adler '14)

Let $\mathcal{C}$ be a class of graphs closed under subgraphs.
The following conditions are equivalent.

1. $\mathcal{C}$ is nowhere dense.
2. $\mathcal{C}$ is stable.
3. $\mathcal{C}$ is dependent, i.e. it has the NIP.

Stability and the NIP are naturally defined on general classes of graphs, not just those closed under taking subgraps.

Hence, stable classes may be a good candidate to study.

Ladders in Graphs

Definition.

Let $\mathcal{C}$ be a class of graphs.

A first-order formula $\varphi(\bar{x}, \bar{y})$ has the *order property* with respect to $\mathcal{C}$ if for every $n \geq 0$ there exist a graph $G \in \mathcal{C}$ and tuples $\overline{a_1} \dots \overline{a_n}$ and $\bar{b}_i \dots \bar{b}_n$ such that

$$G \vDash \varphi[\bar{a}_i, \bar{b}_i] \text{ if, and only if, } i < j.$$

$\mathcal{C}$ is *stable* if there is no such formula with respect to $\mathcal{C}$.

$\overline{b_n}$ $\overline{a_n}$

Stability means that graphs in $\mathcal{C}$ do not

contain definable ladders of large order.

$\overline{b_1}$ $\overline{a_1}$

Stable Classes of Graphs

Definition.

Let $\mathcal{C}$ be a class of graphs.

A first-order formula $\varphi(\bar{x}, \bar{y})$ has the *order property* with respect to $\mathcal{C}$ if for every $n \geq 0$ there exist a graph $G \in \mathcal{C}$ and tuples $\overline{a_1} \dots \overline{a_n}$ and $\bar{b}_i \dots \bar{b}_n$ such that

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$\mathcal{C}$ is *stable* if there is no such formula with respect to $\mathcal{C}$.

$\overline{b_n}$ $\overline{a_n}$

Examples.

- *Nowhere dense classes*
  (Easy: the class of ladders is not nowhere dense.
  But this needs to be shown for definable ladders)
- *The class of cliques*

Non-Examples.

- *Classes of graphs of bounded clique width.*

$\overline{b_1}$ $\overline{a_1}$

Regularity in Stable Classes of Graphs

Theorem. (Malliaris, Shelah '11)

Let $\mathcal{C}$ be a stable class of graphs. Let $G \in \mathcal{C}$.

Then for every $\epsilon > 0$ there is an m such that for all (large enough) $A \subseteq V(G)$:

there is a partition $A_1, \ldots, A_s$ of A into at most m pieces such that

1. $\big||A_i| - |A_j|\big| \leq 1$ for all $i, j \leq s$
2. all pairs (A_i, A_j) are (ϵ, ϵ)-uniform
3. all pieces A_i are ϵ-excellent
4. m is not too big if ϵ is small (a polynomial in ϵ and $\frac{1}{\epsilon}$).

(ϵ, ϵ)-uniform: all except $\epsilon|A_i|$ many vertices in A_i

- have all but at most $\epsilon|A_j|$ many vertices of A_j as neighbours or
- all but at most $\epsilon|A_j|$ many vertices of A_j as neighbours

Indiscernibles

A key technical tool in these results are indiscernible sequences.

Definition.

Let G be a graph and let Δ be a set of formulas.

A sequence $(a_1, \dots, a_l)$ of vertices of G is Δ-indiscernible if for every formula $\varphi(x_1, \dots, x_k) \in \Delta$ and any two increasing sequences $1 \leq i_1 < \dots < i_k \leq l$ and $1 \leq j_1 < \dots < j_k \leq l$ we have

$$G \vDash \varphi[a_{i_1}, \dots, a_{i_k}] \Leftrightarrow G \vDash \varphi[a_{j_1}, \dots, a_{j_k}].$$

Theorem.

Let $\mathcal{C}$ be a stable class of graphs and let Δ be a finite set of FO formulas.

There is a polynomial $p(x)$ such that for all $G \in \mathcal{C}$ and all $m > 0$ and every sequence $(a_1, \dots, a_{p(m)})$ of vertices of G there exists a subsequence $(v_{i_1}, \dots, v_{i_m})$ which is Δ-indiscernible.

This sequence can be computed in time $O\left(|\Delta| \cdot n^{c(|\Delta|)}\right)$.

Uniformly Quasi-Wideness

Definition.

A class $\mathcal{C}$ of graphs is uniformly quasi-wide if
for all $r > 0$ there is $s(r) > 0$ and
all $m > 0$ there is $N(r, m) > 0$ such that
for all $G \in \mathcal{C}$ and all $A \subseteq V(G)$ with $|A| \geq N(r, m)$
there is a set $S \subseteq V(G)$ with $|S| \leq s(r)$ and
$B \subseteq A$ of order $|B| = m$ such that B is $2r$-independent in $G - S$.

Theorem. (Nesetril, Ossona de Mendez)

A class of graphs closed under taking subgraphs is uniformly quasi wide if, and only if, it is nowhere dense.

Theorem. (Dawar, K. 09)

The dominating set problem is fpt on nowhere dense classes.

Problem. The bounds on s, N are huge, as they come from iterated Ramsey.

Uniformly Quasi-Wideness

Theorem. (K., Rabinovich, Siebertz 17)

Let $\mathcal{C}$ be a nowhere dense class of graphs.

For every $r \in \mathbb{N}$ there exists a polynomial $p_r(x)$ and a constant $s(r)$ such that for all m ∈ $\mathbb{N}$ and

for all $G \in \mathcal{C}$, all sets $A \subseteq V(G)$ of size at least $p_r(m)$ there is

a set $S \subseteq V(G)$ of size at most $s(r)$ such that there is a set $B \subseteq A$ of size at least m which is r-independent in $G - S$.

Furthermore, if $K_c \not\preccurlyeq_r G$ for all $G \in \mathcal{C}$, then $s(r) \leq c \cdot r$.

Also, S and B can be computed efficiently.

Corollary.

A class $\mathcal{C}$ of graphs is uniformly quasi-wide with margins $N\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $s : \mathbb{N} \to \mathbb{N}$ if, and only if, it is uniformly quasi-wide with a polynomial margin $N'(r,m) \leq p_r(m)$ and a linear margin $s'(r) \leq c \cdot r$.

Uniformly Quasi-Wideness

Theorem. (K., Rabinovich, Siebertz 17)

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For every $r \in \mathbb{N}$ there exists a polynomial $p_r(x)$ and a constant $s(r)$ such that for all m ∈ $\mathbb{N}$ and

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Furthermore, if $K_c \npreceq_r G$ for all $G \in \mathcal{C}$, then $s(r) \leq c \cdot r$.

Also, S and B can be computed efficiently.

Theorem.

Let $\mathcal{C}$ be a nowhere dense class of graphs.

Then there is a polynomial $p(x)$, a constant α and an algorithm running in time $2^{p(k)} \cdot n^{\alpha}$ which, given an n-vertex graph G and a number k as input, decides whether G contains a connected dominating set of size k.

Polynomial Kernels for Dominating Sets

Theorem.

Let $\mathcal{C}$ be a nowhere dense class of graphs.

For every r the *Distance – r – Dominating Set* problem has a polynomial kernel on $\mathcal{C}$.

(See Sebastian's talk later today)

Theorem. (Berlin+Warsaw 17)

Graphs in stable classes have bounded depth modular decompositions.

Stability and the NIP

Theorem. (Adler, Adler '14)

Let $\mathcal{C}$ be a class of graphs closed under subgraphs.
The following conditions are equivalent.

1. $\mathcal{C}$ is nowhere dense.
2. $\mathcal{C}$ is stable.
3. $\mathcal{C}$ is dependent, i.e. it has the NIP.

Stability and the NIP are naturally defined on general classes of graphs, not just those closed under taking subgraps.

Hence, stable classes may be a good candidate to study.

Conclusion

Interpretations.

- We have seen interpretations in bounded degree graphs.
- Can we characterise classes interpretable in other sparse classes?
- What about classes interpretable in nowhere dense classes of graphs?

Stable graph classes.

- Seem to be a very promising type of graph classes
- Could be a good candidate for FO model checking
- But before that, we need a lot more tools and results for stable classes.

Finally ...

… Happy Birthday, Patrice.