Graph Limits - Finite Forcibility and Computability

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Workshop on Algorithms, Logic and Structure

Graph limits

- Approximate asymptotic properties of large graphs
- Extremal combinatorics/computer science :
  flag algebra method, property testing
  large networks, e.g. the internet, social networks...
- The 'limit' of a convergent sequence of graphs
  is represented by an analytic object called a graphon

Dense graph convergence

- Convergence for dense graphs ($|E| = \Omega(|V|^2)$)
- $d(H, G)$ = probability $|H|$-vertex subgraph of G is H
- A sequence $(G_n)_{n\in\mathbb{N}}$ of graphs is convergent if $d(H, G_n)$ converges for every H

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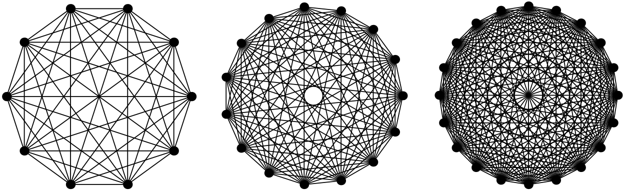

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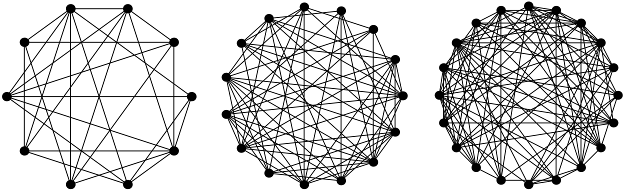

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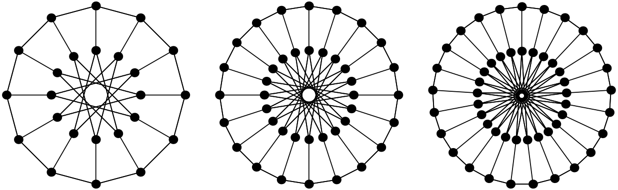

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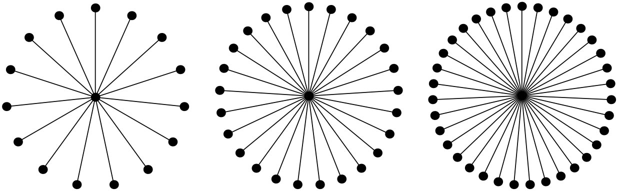

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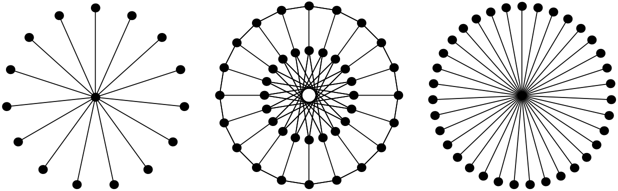
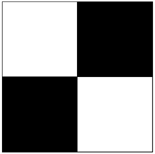
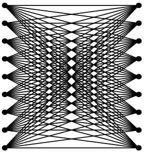
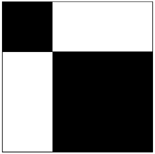


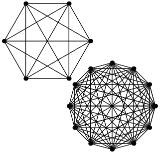
Limit object: graphon

- Graphon: measurable function $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)\ \forall x,y \in [0,1]$
- W-random graph of order n: n random points $x_i \in [0,1]$, edge probability $W(x_i, x_j)$









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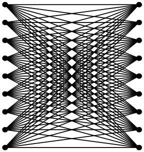
- W is a limit of $(G_n)_{n\in\mathbb{N}}$ if $d(H,W) = \lim\limits_{n\to\infty} d(H,G_n)\ \forall\ H$
  - Every convergent sequence of graphs has a limit
  - W-random graphs converge to W with probability one

Examples of graph limits

- The sequence of complete bipartite graphs, $(K_{n,n})_{n\in\mathbb{N}}$



$$\begin{matrix}
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0&0&0&0&0&0&1&1&1&1&1&1\\
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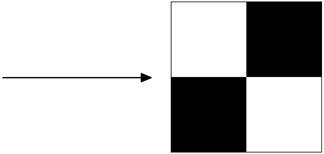
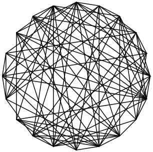
\end{matrix}$$



- The sequence of random graphs, $(G_{n,1/2})_{n\in\mathbb{N}}$



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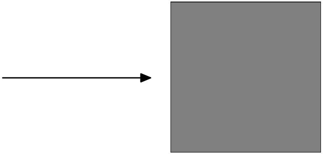
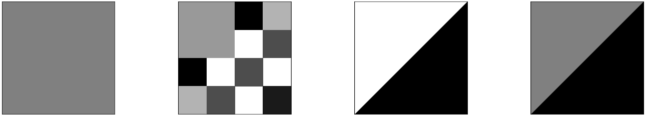


Finitely forcible graphons

- A graphon W is finitely forcible if $\exists\ H_1 \dots H_k$ s.t $d(H_i, W') = d(H_i, W) \implies d(H, W') = d(H, W)\ \forall\ H$



1 2 3 4

1. Thomason (87), Chung, Graham and Wilson (89)
2. Lovász and Sós (2008)
3. Diaconis, Holmes and Janson (2009)
4. Lovász and Szegedy (2011)

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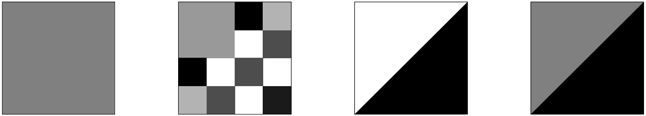

- Every finitely forcible graphon is the unique minimiser of $\sum \alpha_j d(H'_j, W)$, for some α_j and H'_j.
- Conjecture (Lovász and Szegedy)
  Every extremal problem $\min \sum \alpha_j d(H_j, W)$ has a finitely forcible optimal solution.

Simple structure?

- Conjecture (Lovász and Szegedy, 2011)
  The space of typical vertices of a finitely forcible graphon is compact.

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Simple structure?

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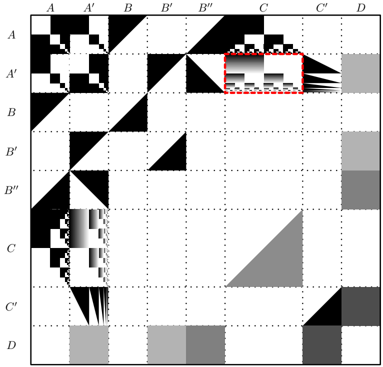
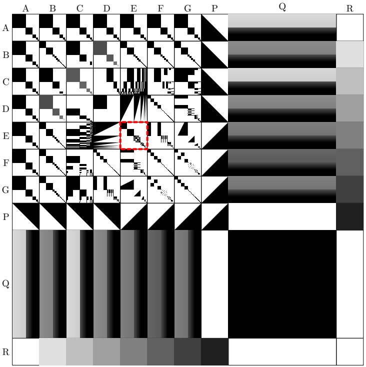
  - Theorem (Glebov, Král', Volec, 2013)
    $T(W)$ can fail to be locally compact
- Conjecture (Lovász and Szegedy, 2011)
  The space of typical vertices of a finitely forcible graphon is finite dimensional.
  - Theorem (Glebov, Klimošová, Král', 2014)
    $T(W)$ can have a part homeomorphic to $[0,1]^\infty$
  - Theorem (Cooper, Kaiser, Král', Noel, 2015)
    $\exists$ finitely forcible W such that every ε-regular partition has at least $2^{\varepsilon^{-2}/\log\log\varepsilon^{-1}}$ parts (for inf. many $\varepsilon \to 0$).

Previous Constructions



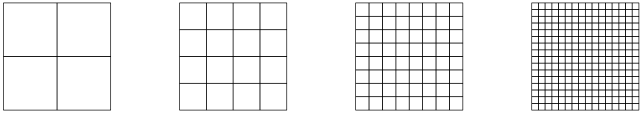


Computability and graphons

- Graphons computable by an algorithm
- Let A_W be the set of all $[s, t, d, p, q] \in \mathbb{N}^5$ s.t.

$$\int_{\left[\frac{s}{2^d}, \frac{s+1}{2^d}\right] \times \left[\frac{t}{2^d}, \frac{t+1}{2^d}\right]} W(x, y)\, dx\, dy \leq \frac{p}{q}$$

- A graphon W is recursive if the set A_W is recursive, i.e. $\mathbb{1}_{A_W}$ is a computable function



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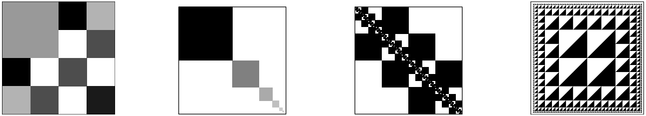


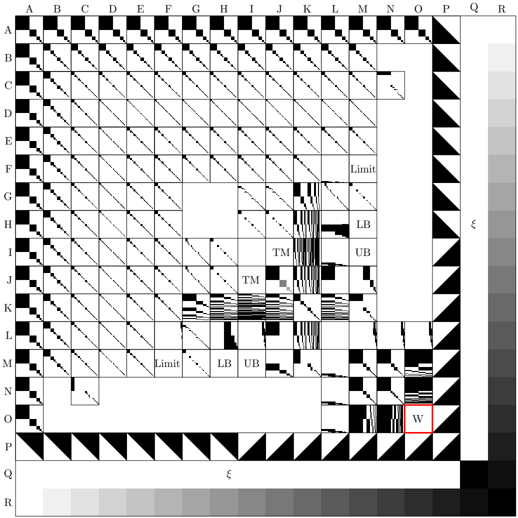
Universal Construction Theorem

- Theorem (Cooper, Král', M.)
  Every computable graphon is a subgraphon of a finitely forcible graphon.
  - Existence of a finitely forcible graphon that is non-compact, infinite dimensional, etc
  - If $f : \mathbb{N} \to \mathbb{N}$ is a recursive function tending to ∞, $\exists$ finite forcible W such that every ϵ-regular partition has at least $2^{\epsilon^{-2}/f(\epsilon^{-1})}$ parts (for inf. many $\epsilon \to 0$)

Ingredients of the proof

- Partitioned graphons
  - vertices with only finitely many degrees
  - parts with vertices of the same degree
- Decorated constraints
  - method for constraining partitioned graphons
  - density constraints rooted in the parts
  - based on notions related to flag algebras
- Simulation of infinitely many Turing machines
  forcing W by fixing its density in dyadic subsquares

Universal construction



Thank you for your attention!