Euler systems and the Bloch–Kato Conjecture (Alpbach Summer School 2021 notes)

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Introduction

The theory of Euler systems is one of the most powerful tools available for studying the arithmetic of global Galois representations. However, constructing Euler systems is a difficult problem, and the list of known constructions was until recently accordingly rather short. In these lecture notes, we outline a general strategy for constructing new Euler systems in the cohomology of Shimura varieties: these Euler systems arise via pushforward of certain units on subvarieties.

We study in detail the Euler system of Beilinson–Flach elements, where the underlying Shimura variety is the fibre product of two modular curves

The lecture notes are structured as follows.

- In Chapter 1, we introduce L-functions and Selmer groups attached to global p-adic Galois representations, and we state the Bloch–Kato conjecture. We also define Euler systems, and we explain their arithmetic applications to the Bloch–Kato conjecture.
- In Chapter 2, we introduce motivic cohomology as a tool for constructing global cohomology classes for Galois representations arising from geometry. We illustrate this theory by some examples, assuming the existence of a supply of subvarieties of appropriate codimension and units on them.
- In Chapter 3, we introduce Siegel units, which are the basic input to many Euler system constructions. We then describe the construction of the Beilinson–Flach Euler system, attached to pairs of modular forms of weight 2. We motivate this construction by explaining its relation to the Rankin–Selberg integral formula.
- In Chapter 4, we discuss the question of proving the non-vanishing of the Euler system of Beilinson–Flach elements. We introduce syntomic cohomology and Fontaine's theory of rings of periods, which we use to relate the evaluation of the Euler system under the syntomic regulator to certain cup products in coherent cohomology. These cup products closely resemble the Rankin–Selberg integral formula and can be interpreted as values of a p-adic L-function.

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CHAPTER 1

Galois representations and Galois cohomology

References: for §§1.1—1.3, an excellent source is Bellaiche's CMI notes on the Bloch–Kato conjecture.

1.1. Galois representations

1.1a. Definitions. Let K be a number field, $\overline{K}$ its algebraic closure, $G_K = \mathrm{Gal}(\overline{K}/K)$; and let p be a prime, and E a finite extension of $\mathbf{Q}_p$. We're interested in representations of G_K on finite-dimensional E-vector spaces V.

We always assume that

1. $\rho : G_K \to \mathrm{Aut}(V) \cong \mathrm{GL}_d(E)$ is continuous (where $d = \dim(V)$), with respect to profinite topology of G_K and the p-adic topology on $\mathrm{GL}_d(E)$.
2. V is "unramified almost everywhere": for all but finitely many prime ideals v of K, we have $\rho(I_v) = \{1\}$, where I_v is an[1] inertia group at v.

1.1b. Examples.

The representation $\mathbf{Z}_p(1)$*.* Let $\mu_{p^n} = \{x \in \overline{K}^\times : x^{p^n} = 1\}$. Then μ_{p^n} is finite cyclic of order p^n and G_K acts on it.

The p-power map sends $\mu_{p^{n+1}} \to \mu_{p^n}$ and we define

$$\mathbf{Z}_p(1) \coloneqq \varprojlim_n \mu_{p^n}, \quad \mathbf{Q}_p(1) \coloneqq \mathbf{Z}_p(1) \otimes \mathbf{Q}_p.$$

This is a 1-dimensional continuous $\mathbf{Q}_p$-linear representation, unramified outside the primes dividing p; G_K acts by "cyclotomic character" $\chi_{\mathrm{cyc}} : G_K \to \mathbf{Z}_p^\times$.

(Notation: for any V, $n \in \mathbf{Z}$, we set $V(n) = V \otimes \mathbf{Q}_p(1)^{\otimes n}$.)

Tate modules of elliptic curves. If A/K is an elliptic curve, then $A(\overline{K})$ is a finitely generated abelian group with a continous G_K-action. Let $A(\overline{K})[p^n]$ denote the subgroup of p^n-torsion points.

Define the *p-adic Tate module*

$$T_p(A) \coloneqq \varprojlim_n A(\overline{K})[p^n] \text{ (w.r.t. multiplication-by-}p\text{ maps)}, \qquad V_p(A) \coloneqq T_p(A) \otimes \mathbf{Q}_p.$$

This is a 2-dimensional continuous G_K-representation, unramified outside the set $\{v : v \mid p\} \cup \{v : A$ has bad reduction at $v\}$. (The same works for abelian varieties of any dimension g, giving $2g$-dimensional representations of G_K.)

Etale cohomology. Let X/K be a smooth algebraic variety. We can define vector spaces

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \quad \text{for } 0 \le i \le 2 \dim X,$$

which are finite-dimensional p-adic Galois representations, unramified outside p and primes of bad reduction[2] of X.

[1] I_v depends on a choice of prime of $\overline{K}$ above v, but only up to conjugation in G_K, so whether or not V is unramified at v is well-defined.

[2] This is a little delicate to define properly if we don't assume X to be proper over K. Formally, we say X has "good reduction" at v if it's isomorphic to the complement of a relative normal crossing divisor in a smooth proper $O_{K,v}$-scheme.

1.1c. Representations coming from geometry. Our second example is a special case of the third: for an elliptic curve A, it turns out that we have $V_p(A) \cong H^1_{\text{ét}}(A_{\overline{K}}, \mathbf{Q}_p)(1)$.

Definition. *We say an E-linear Galois rep V* comes from geometry *if it is a subquotient of*

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)(j) \otimes_{\mathbf{Q}_p} E,$$

for some variety X/K and some integers i, j.

So all of the examples above come from geometry. In these lectures we're only ever going to be interested in representations coming from geometry.

Remark. Conjecturally the representations coming from geometry should be exactly those which are continuous, unramified almost everywhere, and *potentially semistable* at the primes above p (a technical condition from p-adic Hodge theory). This is called the **Fontaine–Mazur conjecture**. ◇

1.2. L-functions of Galois representations

1.2a. Local Euler factors. Let V as above, v unramified prime. Then $\rho(\mathrm{Frob}_v)$ is well-defined up to conjugacy, where Frob_v is the arithmetic Frobenius.

Definition. *The* local Euler factor *of V at v is the polynomial*

$$P_v(V, t) := \det(1 - t \cdot \rho(\mathrm{Frob}_v^{-1})) \in E[t].$$

Examples:

V	$P_v(V,t)$	
$\mathbf{Q}_p$	$1 - t$	
$\mathbf{Q}_p(n)$	$1 - \frac{t}{q_v^n}$,	$q_v = \#\mathbf{F}_v$
$H^1(A_{\overline{K}}, \mathbf{Q}_p)$	$1 - a_v(A)t + q_v t^2$,	$a_v(A) := 1 + q_v - \#A(\mathbf{F}_v)$

1.2b. Global L-functions (sketch). Assume V comes from geometry, and V is semisimple (direct sum of irreducibles). Then $P_v(V,t)$ has coefficients in $\overline{\mathbf{Q}}$ (Deligne); and there is a way of defining $P_v(V,t)$ for bad primes v (case $v \mid p$ is hardest).

Fix an embedding $\iota : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. Then we consider the product

$$L(V, s) := \prod_{v \text{ prime}} P_v(V, q_v^{-s})^{-1}.$$

Miraculously, this converges for $\Re(s) \gg 0$.

E.g. for $V = \mathbf{Q}_p(n)$ this is $\zeta_K(s+n)$, where ζ_K is the Dedekind zeta function of K (which is just the Riemann zeta for $K = \mathbf{Q}$). For $V = H^1(A_{\overline{K}}, \mathbf{Q}_p)$, A an elliptic curve, it is the Hasse–Weil L-function $L(A/K, s)$.

Conjecture 1. *For V semisimple and coming from geometry, $L(V,s)$ has meromorphic continuation to $s \in \mathbf{C}$ with finitely many poles, and satisfies a functional equation relating $L(V,s)$ and $L(V^*, 1-s)$.*

Note that if V is semisimple and comes from geometry, the same is true[3] of V^*, so the conjecture is well-posed. This conjecture is of course super-super-hard – the only cases where it is known is where we can relate V to something *automorphic*, e.g. a modular form.

There are lots of conjectures (and a rather smaller set of theorems) relating properties of arithmetic objects to values of their L-functions; the Birch–Swinnerton-Dyer conjecture is perhaps the best-known of these. As we've just seen, all the information about an elliptic curve you need to define its L-function is encoded in the Galois action on its Tate module; so can we express the BSD conjecture purely in terms of Galois representations? This will be the topic of the next section.[4]

[3] It is not obvious if this holds without the semisimplicity assumption.

[4] Actually the answer is "no, we can't" – as far as I'm aware, there is no purely Galois-representation-theoretic statement that is precisely equivalent to BSD. But we can get pretty close, as we'll shortly see.

1.3. Galois cohomology

1.3a. Setup. There is a cohomology theory for Galois representations[5]: for V an E-linear G_K-rep, we get E-vector spaces $H^i(K,V)$, zero unless $i = 0,1,2$. Mostly we care about H^0 and H^1, which are given as follows

$$H^0(K,V) = V^{G_K}$$
$$H^1(K,V) = \frac{\{\text{cts fcns } s : G_K \to V \text{ such that } s(gh) = s(g) + gs(h)\}}{\{\text{fcns of the form } s(g) = gv - v \text{ for some } v \in V\}}.$$

These are well-behaved: short exact sequences of V's give long exact sequences of cohomology, for instance. Unfortunately they're *not* finite-dimensional in general.

1.3b. The Kummer map. For $V = \mathbf{Q}_p(1)$ the Galois cohomology is related to the multiplicative group K^*. To see this, we have to first think a bit about cohomology with *finite* coefficients.

For any n, we have a short exact sequence

$$0 \longrightarrow \mu_{p^n} \longrightarrow \overline{K}^\times \xrightarrow{[p^n]} \overline{K}^\times \longrightarrow 0$$

which leads to a long exact sequence

$$0 \longrightarrow \mu_{p^n}^{G_K} \longrightarrow K^\times \xrightarrow{[p^n]} K^\times \longrightarrow H^1(K, \mu_{p^n})$$

and thus an injection[6]

$$K^\times \otimes \mathbf{Z}/p^n\mathbf{Z} \hookrightarrow H^1(K, \mu_{p^n}).$$

Passing to the inverse limit we get a map (**Kummer map**)

$$\kappa_p : K^\times \otimes \mathbf{Z}_p \hookrightarrow H^1(K, \mathbf{Z}_p(1)) \quad \text{or} \quad K^\times \otimes \mathbf{Q}_p \hookrightarrow H^1(K, \mathbf{Q}_p(1)).$$

Remark. This already shows that $H^1(K, \mathbf{Q}_p(1))$ can't be finite-dimensional, because $K^\times$ has countably infinite rank. ◇

The same argument works for elliptic curves: we get an embedding

$$E(K) \otimes \mathbf{Q}_p \hookrightarrow H^1(K, V_p(E)).$$

1.3c. Selmer groups. Since the groups $H^1(K,V)$ can be infinite-dimensional, it's useful to "cut down to size" by imposing extra conditions on our H^1 elements. We'll do this by localising at primes of K. Note that we have maps

$$H^i(K,V) \to H^i(K_v,V) \text{ for all primes } v,$$

and the local groups $H^i(K_v,V)$ are finite-dimensional.

Definition. *A* local condition *on V at prime v is an E-linear subspace $\mathcal{F}_v \subseteq H^1(K_v,V)$.*

Examples:

- *strict* local condition $\mathcal{F}_{v,\mathrm{strict}} = \{0\}$
- *relaxed* local condition $\mathcal{F}_{v,\mathrm{rel}} =$ all of $H^1(K_v,V)$
- *unramified* local condition (usually only interesting for $v \nmid p$)

$$\mathcal{F}_{v,\mathrm{ur}} = \text{image}\left(H^1(G_{K_v}/I_v, V^{I_v}) \to H^1(K_v,V)\right)$$

- *Bloch–Kato "finite"* local condition $\mathcal{F}_{v,\mathrm{BK}}$ (for $v \mid p$) – defined using p-adic Hodge theory.[7]

[5] Technical point: our representations are all continuous, so we shall work with cohomology defined by continuous cochains, which is slightly different from the cohomology of G_K as an abstract group.

[6] In fact this is an isomorphism, because $H^1(K, \overline{K}^\times)$ is zero ("Hilbert's theorem 90")

[7] We will see the precise definition in Section 4.1c.

Definition. *A* Selmer structure *is a collection* $\mathcal{F} = (\mathcal{F}_v)_{v \text{ prime of } K}$*, satisfying the following condition: for almost all* v *we have* $\mathcal{F}_v = \mathcal{F}_{v,\mathrm{ur}}$*. If* $\mathcal{F}$ *is a Selmer structure we define the corresponding Selmer group by*

$$\mathrm{Sel}_{\mathcal{F}}(K, V) = \{x \in H^1(K, V) : \mathrm{loc}_v(x) \in \mathcal{F}_v \ \forall v\}.$$

Theorem 2 (Tate). *For any Selmer structure* $\mathcal{F}$*, the space* $\mathrm{Sel}_{\mathcal{F}}(K, V)$ *is finite-dimensional over* $\mathbf{Q}_p$*.*

Sketch of proof. It's easy to see that if this statement is true for one $\mathcal{F}$, it's true for any $\mathcal{F}$, since the *local* Galois cohomology groups $H^1(K_v, V)$ are all finite-dimensional. We now choose a particular Selmer structure $\mathcal{F}$ (exercise: which?) such that $\mathrm{Sel}_{\mathcal{F}}(K, V)$ is the image of the map

$$H^1\left(\mathrm{Gal}(K^\Sigma/K), V\right) \hookrightarrow H^1(K, V),$$

where K^Σ is the maximal extension of K unramified outside some finite set of places Σ containing all infinite places, all places above p, and all places where V is ramified. This reduces us to what Tate actually proved, which is that the cohomology groups of $\mathrm{Gal}\left(K^\Sigma/K\right)$ are finite-dimensional. □

We're mostly interested in three specific choices of Selmer structure, differing only in the choices of the $\mathcal{F}_v$ at primes $v \mid p$: we define the *strict Selmer group* $\mathrm{Sel}_{\mathrm{strict}}(K, V)$ by taking $\mathcal{F}_v = \mathcal{F}_{v,\mathrm{ur}}$ for $v \nmid p$, and $\mathcal{F}_v = \mathcal{F}_{v,\mathrm{strict}}$ for $v \mid p$; and similarly the *relaxed Selmer group* and the *Bloch–Kato Selmer group*.

Hence the strict and Bloch–Kato Selmer groups satisfy

$$\mathrm{Sel}_{\mathrm{strict}}(K, V) \subseteq \mathrm{Sel}_{\mathrm{BK}}(K, V) \subseteq \mathrm{Sel}_{\mathrm{rel}}(K, V).$$

Remark. As will soon become clear, it is $\mathrm{Sel}_{\mathrm{BK}}(K, V)$ which is the most important of all. We care about $\mathrm{Sel}_{\mathrm{strict}}(K, V)$ and $\mathrm{Sel}_{\mathrm{rel}}(K, V)$ because they are easier to study, and they give us upper and lower bounds for the thing we care about. ◇

Example. Recall that for $V = \mathbf{Q}_p(1)$ we had the Kummer map

$$K^\times \otimes \mathbf{Q}_p \hookrightarrow H^1(K, \mathbf{Q}_p(1)).$$

One can check that this induces an isomorphism

$$\mathcal{O}_K^\times \otimes \mathbf{Q}_p \xrightarrow{\cong} \mathrm{Sel}_{\mathrm{BK}}(K, \mathbf{Q}_p(1)),$$

and similarly

$$\mathcal{O}_K[1/p]^\times \otimes \mathbf{Q}_p \xrightarrow{\cong} \mathrm{Sel}_{\mathrm{rel}}(K, \mathbf{Q}_p(1)).$$

The strict Selmer group, on the other hand, should be zero, but we can't prove this: it's exactly Leopoldt's conjecture for K. ◇

1.3d. The Bloch–Kato conjecture. Let V be a representation coming from geometry.

Conjecture 3 (Bloch–Kato). *We have*

$$\dim \mathrm{Sel}_{\mathrm{BK}}(K, V) - \dim H^0(K, V) = \mathrm{ord}_{s=0} L(V^*(1), s).$$

There are refined versions using $\mathbf{Z}_p$-modules in place of $\mathbf{Q}_p$-vector spaces, which predict the leading term of the L-function up to a unit; but we won't go into these here.

Let's look at what the conjecture says in some example cases.

Example 1: $V = \mathbf{Q}_p$. Here $L(V^*(1), s) = L(\mathbf{Q}_p, s+1) = \zeta_K(s+1)$, so the right-hand side is the order of vanishing of $\zeta_K(s)$ at $s = 1$, which is -1 for every K (there's a simple pole). The left-hand side is $\dim \mathrm{Sel}_{\mathrm{BK}}(K, \mathbf{Q}_p) - 1$, so the conjecture predicts that $\mathrm{Sel}_{\mathrm{BK}}(K, \mathbf{Q}_p) = 0$.

Exercise: Prove this. You'll need to use the finiteness of the ideal class group of K, together with the fact that for this representation the local condition $\mathcal{F}_{v,\mathrm{BK}}$ agrees with $\mathcal{F}_{v,\mathrm{ur}}$ for primes $v \mid p$.

Example 2: $V = \mathbf{Q}_p(1)$. Here $L(V^*(1), s) = \zeta_K(s)$. Inspecting the functional equation for Dedekind zeta functions, we see that $\operatorname{ord}_{s=0} \zeta_K(s) = r_1 + r_2 - 1$, where r_1, r_2 are the numbers of real and complex places respectively. (In particular, if $K = \mathbf{Q}$, then $\zeta(0) = -\frac{1}{2}$ is finite and non-zero.) On the algebraic side, we have $H^0(K, \mathbf{Q}_p(1)) = 0$ and

$$\dim \mathrm{Sel}_{\mathrm{BK}}(K, V) = \dim_{\mathbf{Q}_p} \left(\mathcal{O}_K^\times \otimes \mathbf{Q}_p\right) = \operatorname{rank} \mathcal{O}_K^\times.$$

So the Bloch–Kato conjecture here is exactly Dirichlet's unit theorem.

Example 3: Elliptic curves. If V is $V_p(E)$ for an elliptic curve E, then:

- the H^0 term is zero;
- the Kummer map lands inside the BK Selmer group, and gives an embedding
$$E(K) \otimes \mathbf{Q}_p \hookrightarrow \mathrm{Sel}_{\mathrm{BK}}(K, V),$$
so that $\dim \mathrm{Sel}_{\mathrm{BK}} \ge \operatorname{rank}(E/K)$, with equality iff the p-part of Sha is finite;
- $\operatorname{ord}_{s=0} L(V^*(1), s) = \operatorname{ord}_{s=1} L(E/K, s)$.

So this instance of Bloch–Kato is closely related to (but not quite the same as) the Birch–Swinnerton-Dyer conjecture.

Remark. Notice that $L(V^*(1), s)$ is expected to be related to $L(V, -s)$ via a functional equation; but this functional equation will involve various Γ functions as factors, which can have poles, so the orders of vanishing of the two functions at 0 are not the same in general, as we saw for $\mathbf{Q}_p$ and $\mathbf{Q}_p(1)$. On the Selmer-group side there's a corresponding relation between $\mathrm{Sel}_{\mathrm{BK}}(K, V)$ and $\mathrm{Sel}_{\mathrm{BK}}(K, V^*(1))$ coming from the Poitou–Tate global duality theorem in Galois cohomology. One can check that these factors precisely cancel out: if $L(V, s)$ has a functional equation of the expected form, then the Bloch–Kato conjecture holds for $V^*(1)$ if and only if it holds for V. This is a wonderful (but rather involved) exercise. $\diamond$

1.4. Euler systems

We'll now introduce the key subject of these lectures: Euler systems, which are tools for studying and controlling Selmer groups. In this section we'll give the abstract definition of an Euler system, and explain (without proofs) why the existence of an Euler system for some Galois representation has powerful consequences for Selmer groups.

References: The standard work on this topic is Karl Rubin's orange book *Euler Systems* [**Rub00**]. There are also two alternative accounts in Rubin's 2004 Park City lecture notes, and in the book *Kolyvagin Systems* [**MR04**] by Mazur and Rubin.

1.4a. The definition.

Let:

- V a $G_{\mathbf{Q}}$-representation (for simplicity)
- $T \subset V$ a $G_{\mathbf{Q}}$-stable $\mathbf{Z}_p$-lattice
- Σ a finite set of primes containing p and all ramified primes for V

Since V is a $G_{\mathbf{Q}}$-rep, we can consider it as a G_K-rep for any number field K and form $H^i(K, V)$, and there are **corestriction** or **norm** maps

$$\mathrm{norm}_K^L : H^i(L, V) \to H^i(K, V) \quad \text{if } L \supset K.$$

If K is Galois, $H^i(K, V)$ is a module over $\mathbf{Q}_p[\mathrm{Gal}(K/\mathbf{Q})]$. Similarly for cohomology of lattices $H^i(K, T)$.

Definition. *An* Euler system for (T, Σ) *is a collection* $\mathbf{c} = (c_m)_{m \ge 1}$, *where* $c_m \in H^1(\mathbf{Q}(\mu_m), T)$, *satisfying the following compatibility for any* $m \ge 1$ *and* ℓ *prime:*

$$\mathrm{norm}_{\mathbf{Q}(\mu_m)}^{\mathbf{Q}(\mu_{m\ell})}(c_{m\ell}) = \begin{cases} c_m & \text{if } \ell \in \Sigma \text{ or } \ell \mid m \\ P_\ell(V^*(1), \sigma_\ell^{-1}) \cdot c_m & \text{otherwise} \end{cases}$$

where σ_ℓ *is the image of* Frob_ℓ *in* $\mathrm{Gal}(\mathbf{Q}(\mu_m)/\mathbf{Q})$. *An* Euler system for V *is an Euler system for* (T, Σ), *for some* $T \subset V$ *and some* Σ.

Intuitively, these are cohomological avatars of an L-function – the norm relation reflects the way we build up $L(V,s)$ as an Euler product.

1.4b. Bounding Selmer groups. The main reason to care about these objects is the following theorem (called the *Euler system machine*), which is due to Rubin [**Rub00**], building on earlier work of Kato [**Kat04**], Kolyvagin [**Kol90**], and Thaine [**Tha88**]:

THEOREM 4. *Suppose* $\mathbf{c}$ *is an Euler system for* (T,Σ) *with* c_1 *non-zero, and suppose* V *satisfies various technical conditions. Then* $\dim \mathrm{Sel}_{\mathrm{rel}}(\mathbf{Q},V) \le \dim(V^{c=-1})$*, where* c *denotes complex conjugation.*

So we are "not far away" from controlling the Bloch–Kato Selmer group (and there are finer versions incorporating more local information at p, which allow us to get at the Bloch–Kato Selmer group itself). This is the key step in many of the known cases of the Bloch–Kato conjecture.

For the purposes of these lectures we don't need to know how this theorem is proved – our goal is to understand how to *build* Euler systems, which is a separate problem. If you do want to know about the proof, then see the references listed above.

REMARK.

- The technical conditions are to do with the image of $G_{\mathbf{Q}}$ in $\mathrm{GL}(V)$. This needs to be "large enough" in a certain precise sense, which in particular implies that V is irreducible.
- For the proof of the theorem, we don't actually need c_m to be defined for all m; it's enough to have c_m for all integers m of the form $p^k m_0$, where $k \ge 0$ and m_0 is a square-free product of primes not in Σ.
- More generally, one can also define Euler systems for G_K-representations, for K a number field. In place of cyclotomic fields, one has to have classes over different ray class fields of K. However, we'll only work with $K = \mathbf{Q}$ here.
- There is also a notion of "anticyclotomic Euler system", which applies when you have a representation V of G_K, a quadratic extension L/K, and cohomology classes for V over the *anticyclotomic extensions* of L, which are the abelian extensions of L such that conjugation by $\mathrm{Gal}(L/K)$ acts on their Galois groups by -1. The most important example of an anticyclotomic Euler system is Kolyvagin's **Euler system of Heegner points** [**Kol90**], where $K = \mathbf{Q}$, $V = V_p(E)$ for E an elliptic curve, and L is an imaginary quadratic field. Other examples of anticyclotomic Euler systems have recently been found by Cornut, and by Jetchev and his collaborators. $\diamond$

1.4c. Cyclotomic units. As a first example, we're going to build an Euler system for $V = \mathbf{Q}_p(1)$. Recall that we have Kummer maps $\kappa_p : K^\times \to H^1(K, \mathbf{Z}_p(1))$. Also, for L/K finite, the Galois corestriction map corresponds to the usual field norm. So we want to find good elements of the multiplicative groups of cyclotomic fields, satisfying compatibilities under the norm maps.

Fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}^\times$ and let $\zeta_m = \iota^{-1}(e^{2\pi i/m}) \in \mu_m$.

DEFINITION. *For* $m > 1$*, set* $u_m = 1 - \zeta_m \in \mathbf{Q}(\mu_m)^\times$*. For all* m *(including* $m = 1$*), set*

$$v_m = \begin{cases} u_m & \text{if } p \mid m, \\ \mathrm{norm}^{\mathbf{Q}(\mu_{pm})}_{\mathbf{Q}(\mu_m)}(u_{pm}) & \text{if } p \nmid m. \end{cases}$$

A pleasant computation (exercise!) shows that $v_1 = p$, and that

$$\mathrm{norm}^{\mathbf{Q}(\mu_{m\ell})}_{\mathbf{Q}(\mu_m)} v_{m\ell} = \begin{cases} v_m & \text{if } \ell \mid m \text{ or } \ell = p, \\ (1 - \sigma_\ell^{-1}) \cdot v_m & \text{otherwise.} \end{cases}$$

THEOREM 5. *The classes* $c_m = \kappa_p(v_m)$ *are an Euler system for* $(\mathbf{Z}_p(1), \{p\})$. $\square$

1.4d. Twisting Euler systems. Recall that $V(n) = V \otimes \mathbf{Q}_p(1)^{\otimes n}$.

Theorem 6 (Soulé twists). *There is a canonical bijection* Tw_n *sending Euler systems for* V *to Euler systems for* $V(n)$*, for any* n*.*

It's important to note that if $\mathbf{d} = \mathrm{Tw}_n(\mathbf{c})$, then we can't find out d_1 just from c_1 (we need to know all the c_{mp^k} for $k \geq 0$). This matters, because we need the 'bottom', $m = 1$ class to be non-zero to apply Theorem 4. Determining the class c_1 of the twisted Euler system is a very deep problem; results describing these classes are called *explicit reciprocity laws.*

Twisting cyclotomic units. What do we get if we twist the cyclotomic-unit Euler system?

For all *odd* n, the n-th twist trivially has $c_1 = 0$ (because the units of $\mathbf{Q}(\mu_m)$ live in the real subfield $\mathbf{Q}(\mu_m)^+$ up to finite index). The even twists are more subtle.

Theorem 7 (Soulé). *For* n *even, the bottom class in the* n*-th twist of the cyclotomic units is non-zero, and is a basis of the 1-dimensional space* $H^1_{\mathrm{rel}}(\mathbf{Q}, \mathbf{Q}(n+1))$*.*

The Bloch–Kato Selmer group depends sharply on whether n is positive or negative. For even $n > 0$, the relaxed Selmer and Bloch–Kato Selmer agree, so $\mathrm{Sel}_{\mathrm{BK}}(\mathbf{Q}, \mathbf{Q}_p(n+1))$ is one-dimensional (consistently with the Bloch–Kato conjecture, since $L(V^*, 1) = \zeta(-n)$ vanishes to degree 1).

For even $n < 0$, the Bloch–Kato Selmer agrees with the *strict* Selmer, and the twisted cyclotomic-unit class is *not* in this space; its localisation at p is related to the algebraic part of $\zeta(-n)$ (the $|n|$-th Bernoulli number). Hence $\mathrm{Sel}_{\mathrm{BK}}(\mathbf{Q}, \mathbf{Q}_p(n+1))$ is zero in this range, consistently with the fact that $\zeta(-n) \neq 0$. So we obtain the Bloch–Kato conjecture for $V = \mathbf{Q}_p(n+1)$ for all even n (and by duality one can obtain the result for all n).

CHAPTER 2

A toolkit for building Euler systems

2.1. Etale cohomology and the Hochschild–Serre spectral sequence

(*References*: not as many as there should be. Jannsen's article "Continuous étale cohomology" [**Jan88**] has some of the details, but it is not an easy read.)

We saw before that, for a variety X/K, the étale cohomology groups $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ were an interesting source of Galois representations.

But this isn't the only thing we can do with étale cohomology. Rather than base-extending to $\overline{K}$, we can also take étale cohomology of X/K directly[1]; there are groups $H^i_{\text{ét}}(X, \mathbf{Q}_p(m))$ for all i and m. These "absolute" étale cohomology groups are *not* themselves Galois representations, but it turns out that these are related to the Galois cohomology of the étale cohomology over $\overline{K}$:

Theorem 8 (Jannsen). *For any variety X/K, and any n, there is a convergent "Hochschild–Serre" spectral sequence*

$$E_2^{ij} = H^i\left(K, H^j_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)(n)\right) \Rightarrow H^{i+j}_{\text{ét}}(X, \mathbf{Q}_p(n)).$$

In particular, we get edge maps $H^i(X, \mathbf{Q}_p(n)) \to H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(n))^{G_K}$, and if F^1H^i denotes the kernel of this map (the "homologically trivial" classes), there is a map

$$F^1 H^i(X, \mathbf{Q}_p(n)) \to H^1\left(K, H^{i-1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)(n)\right).$$

So, if X is defined over $\mathbf{Q}$ and V is the Galois representation $H^{i-1}(X_{\overline{\mathbf{Q}}})$ (or a quotient of it), we can try to construct an Euler system for V by building classes in $F^1H^i(X_{\mathbf{Q}})$, and more generally in $F^1H^i(X_{\mathbf{Q}(\mu_m)})$ for varying m, and then projecting these to V.

How will we do this? We'll use a tool called *motivic cohomology*.

2.2. Motivic cohomology

References: Mazza–Voevodsky–Weibel, *Lecture notes on motivic cohomology* [**MVW06**]; Beilinson, *Higher regulators and values of L-functions* [**Beĭ84**].

2.2a. Definitions. There is a cohomology theory for algebraic varieties called *motivic cohomology*, introduced by Beilinson and greatly refined by the late Vladimir Voevodsky.

Definition. *If X is a regular scheme, we define motivic cohomology groups*

$$H^i_{\text{mot}}(X, \mathbf{Q}(n)) = \mathrm{Gr}^\gamma_n K_{2n-i}(X) \otimes \mathbf{Q},$$

the n-th graded piece of the γ-filtration of the $(2n-i)$-th algebraic K-theory of X.

Voevodsky defined integral versions of these, $H^i_{\text{mot}}(X, \mathbf{Z}(n))$, which have a reasonably concrete description via Bloch's higher Chow groups. These groups vanish for $i > 2n$, or for $n < 0$ (but, rather disturbingly, it is not known whether $H^i_{\text{mot}}(X, \mathbf{Z}(n))$ vanishes for $i < 0$).

[1]Technical point: what we actually want here is "continuous étale cohomology" in the sense of Jannsen. This is consistent with our use of continuous cochains to define cohomology of Galois representations.

For small i and n the motivic cohomology groups have explicit descriptions:

- If X is connected, then $H^0_{\mathrm{mot}}(X, \mathbf{Z}(0)) = \mathbf{Z}$.
- $H^1_{\mathrm{mot}}(X, \mathbf{Z}(1)) \cong \mathcal{O}(X)^\times$.
- (Landsburg [**Lan91**]) If S is an algebraic surface over a field k, then $H^3_{\mathrm{mot}}(S, \mathbf{Z}(2))$ is isomorphic to the quotient
$$\left\{ \begin{array}{c} \text{formal sums} \sum_i (Z_i, u_i),\ Z_i \subset S \text{ irreducible curve,} \\ u_i \in k(Z_i)^\times, \text{ with} \sum_i \operatorname{div} u_i = 0 \end{array} \right\} / \sim$$
where $\sim$ is some equivalence relation (involving K_2 of the function field of S).

Motivic cohomology has good functorial properties, e.g. the following:

- **Cup products**: for a variety X and any i, j, n, m, there are products
$$H^i_{\mathrm{mot}}(X, \mathbf{Q}(m)) \times H^j_{\mathrm{mot}}(X, \mathbf{Q}(n)) \to H^{i+j}_{\mathrm{mot}}(X, \mathbf{Q}(m+n));$$
- **Pushforward maps**: if $\iota : Z \to X$ is a finite morphism (e.g. the inclusion of a closed subvariety), then there are pushforward maps
$$\iota_\star : H^i_{\mathrm{mot}}(Z, \mathbf{Z}(n)) \to H^{i+2c}_{\mathrm{mot}}(X, \mathbf{Z}(n+c)).$$
where c is the codimension of Z. In particular, the pushforward of the identity class $1_Z \in H^0_{\mathrm{mot}}(Z, \mathbf{Z}(0))$ is a class in $H^{2c}_{\mathrm{mot}}(X, \mathbf{Z}(c))$, the *cycle class* of Z.

We can build up elements in $H^i(X, \mathbf{Z}(n))$ for various different values of i and n by combining units, pushforward from subvarieties, and cup-products.

Example. Let X be a smooth variety.

- For any $c \geq 0$, mapping a codimension c subvariety $\iota : Z \hookrightarrow X$ to its cycle class $\iota_\star(1_Z)$ defines an isomorphism beteween $H^{2c}(X, \mathbf{Z}(c))$ and the *Chow group* $\mathrm{CH}^c(X)$ of codimension c cycles up to rational equivalence.
- If $X = S$ is a smooth surface, and $\iota : Z \hookrightarrow S$ is a smooth curve, then for any $u \in \mathcal{O}(Z)^\times$, $\iota_\star(u) \in H^3_{\mathrm{mot}}(S, \mathbf{Z}(2))$ is just the class of (Z, u).

◇

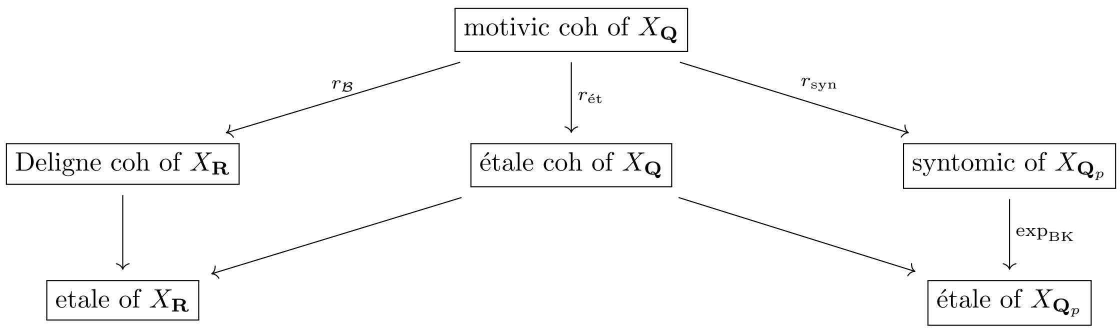
2.2b. Regulators. Crucially, motivic cohomology is related to étale cohomology. If X is a smooth variety over some field K, and p is a prime (nonzero in K), then there are *étale regulator maps*
$$r_{\text{ét}} : H^i_{\mathrm{mot}}(X, \mathbf{Z}(n)) \otimes \mathbf{Z}_p \to H^i_{\text{ét}}(X, \mathbf{Z}_p(n)).$$

Remark. If $X = \operatorname{Spec} K$, then the étale regulator on this group is the Kummer map
$$\kappa_p : \mathcal{O}(X)^\times = K^\times \longrightarrow H^1_{\text{ét}}(K, \mathbf{Z}_p(1)),$$
which we saw before. ◇

This is why motivic cohomology is useful here: it allows us to construct classes in étale cohomology via the map $r_{\text{ét}}$. This is just one of several regulator maps; the others we'll use are:

- the Beilinson regulator into Deligne cohomology when $K = \mathbf{R}$:
$$r_{\mathcal{B}} : H^i_{\mathrm{mot}}(X, \mathbf{Z}(n)) \otimes \mathbf{R} \longrightarrow H^i_{\mathcal{D}}(X, \mathbf{R}(n));$$
- the syntomic regulator into syntomic cohomology, when $K = \mathbf{Q}_p$:
$$r_{\mathrm{syn}} : H^i_{\mathrm{mot}}(X, \mathbf{Z}(n)) \otimes \mathbf{Q}_p \longrightarrow H^i_{\mathrm{syn}}(X, \mathbf{Q}_p(n)).$$
(This will be discussed in more detail in Chapter 4.)

If X is defined over $\mathbf{Q}$, then we can base-extend it to either $\mathbf{C}$ or $\mathbf{Q}_p$, so we get a choice of cohomology theories, and these fit into a rather elaborate diagram:



Here "$\exp_{\text{BK}}$" is the Bloch–Kato exponential map of p-adic Hodge theory[2]. The kernel of this map is well-understood, and its image is, roughly, the local H^1_{f} space giving the local condition for the Selmer group.

Crucially, this diagram is *much less symmetric than it looks.* The Galois group of $\mathbf{R}$ is finite, and hence $H^i(\mathbf{R}, -)$ vanishes for $i > 1$; so the étale cohomology of $X_{\mathbf{R}}$ is just the complex conjugation invariants in Betti cohomology. However, the Galois group of $\mathbf{Q}_p$ is big and interesting! So the lower left corner of the diagram only detects if a class in étale coh lives in F^1 or not, and it says nothing about the higher filtration steps; but the lower right arrow can see these.

2.2c. Games with theories. We're now going to play a rather delicate game with these theories. Suppose now that X is defined over $\mathbf{Q}$, and we want to build Euler systems for V, where V is the Galois representation $H^r_{\text{ét}}(X_{\overline{\mathbf{Q}}})(n)$ (for some r and n), or a direct summand of it. It's the étale cohomology of $X_{\mathbf{Q}}$ we really want to study, since this is related to the Galois cohomology of V. But this is very inexplicit (it's a nightmare to check whether a class is 0 or not).

The other two theories, Deligne and syntomic, are **analytic** in nature, given by complexes of differential forms (real-analytic for Deligne cohohomology, p-adic analytic for syntomic) which are, at least in principle, **computable**; this is fairly easy for Deligne cohomology, and possible – although much more difficult – for syntomic cohomology.

So, we will try to construct an Euler system for V as follows:

- First, we will try to write down classes in the correct group, $H^{r+1}_{\text{mot}}(X, \mathbf{Q}(n))$, using units, pushforward from subvarieties, etc.
- We'll choose a class $\mathbf{z}$ so that $r_{\mathbf{C}}(\mathbf{z})$ can be computed, and is interesting and non-zero (so in particular $\mathbf{z}$ is nonzero).
- We'll rig things so $r_{\mathbf{C}}(\mathbf{z})$ goes to 0 in Betti cohomology of $X_{\mathbf{C}}$. Commutativity of the left diamond now tells us that $r_{\text{ét}}(\mathbf{z})$ is homologically trivial, i.e. lands in the filtration step $F^1 H^{r+1}_{\text{ét}}(X, \mathbf{Q}_p(n))$. This gives us a class $\mathbf{z}_V \in H^1(\mathbf{Q}, V)$.
- The image of this class under $H^1(\mathbf{Q}, V) \xrightarrow{\text{loc}_p} H^1(\mathbf{Q}_p, V)$ is given by

$$\text{loc}_p(\mathbf{z}_V) = \exp_{\text{BK}}(r_{\text{syn}}(\mathbf{z})).$$

  So if we can compute $r_{\text{syn}}(\mathbf{z})$, and show that it's non-zero, we deduce that $\mathbf{z}_V$ isn't zero either.

If we can get this to work, then we have an interesting, nontrivial cohomology class for V; and we can try to build an Euler system on top of this.

Sadly, for a "random" variety X, we get stuck at the first step: it's not clear how to find lots of subvarieties, or lots of units, on X. But we're going to home in on the case where X is a *Shimura variety* – a variety

[2] The commutativity of the right-hand diamond is a very deep theorem, due to Nekovar and Niziol [**NN16**]

coming from automorphic theory, such as a modular curve. Then we can try and write down units and subvarieties using automorphic tools, and feed them into the above machine.

2.3. Numerology

2.3a. For a curve. Suppose C is a curve. Then the interesting Galois representations in étale cohomology live in degree 1: we want $V = H^1_{\text{ét}}(C_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)$, or a direct summand of it. Since we can twist Euler systems, we can choose to work with the twist $V(n)$ for any integer n.

I'm going to suppose C is an *affine* curve[3], which implies $H^2(C_{\overline{\mathbf{Q}}}, \mathbf{Z}_p) = 0$ (Artin vanishing). So the Hochschild–Serre spectral sequence gives us a map

$$H^2_{\text{ét}}(C, \mathbf{Z}_p(n)) \to H^1\left(\mathbf{Q}, H^1_{\text{ét}}(C_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)(n)\right) \to H^1(\mathbf{Q}, V(n)).$$

Using the étale regulator, we can get classes in this group if we can construct motivic classes in $H^2_{\text{mot}}(C, \mathbf{Z}(n))$. How can we build classes in this group geometrically?

- For $n \leq 0$ this is hopeless, because the motivic cohomology is 0 in this range.
- For $n = 1$, you can use cycle classes of codimension 1 subvarieties of C – i.e., points. For C an elliptic curve, this is Kolyvagin's original approach [**Kol90**]: he built an Euler system for elliptic curves using cycle classes of Heegner points. However, this gives an anticyclotomic Euler system (relative to some choice of imaginary quadratic field), not a full Euler system in the sense of §1.4a.
- For $n = 2$, you can use cup-products of units: given units on C, we get classes in $H^1_{\text{mot}}(C, \mathbf{Z}(1))$, and the cup-product of two such classes lands in $H^2_{\text{mot}}(C, \mathbf{Z}(2))$. This is Kato's approach [**Kat04**], when C is an elliptic curve (or more generally a modular curve); the units he used were so-called *Siegel units*, which we will define in the next section.
- $n \geq 3$ can also be made to work (but gives no more information than for $n = 2$).

2.3b. For a surface. Now let's consider a surface S. Here the interesting cohomology of $S_{\overline{\mathbf{Q}}}$ appears in degree 2, so we want to take V a direct summand of $H^2_{\text{ét}}\left(S_{\overline{\mathbf{Q}}}, \mathbf{Q}_p(n)\right)$ for some n. So, in order to construct an Euler system for V, we want to write down classes in $H^3_{\text{mot}}(S, \mathbf{Z}(n))$ for some n. Again, if S is affine, $H^3_{\text{ét}}(S_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(n))$ vanishes and so all classes are homologically trivial.

We need $n \geq 2$, since otherwise the motivic cohomology vanishes. The sensible choices for a twist n are:

(1) $n = 3$: we can get classes here as cup-products $\kappa_p(u_1) \cup \kappa_p(u_2) \cup \kappa_p(u_3)$, where u_1, u_2, u_3 are units on S.
(2) $n = 2$: we can get classes by taking a curve $Z \subset S$ and a unit $u \in \mathcal{O}(Z)^\times$, and considering the pushforward $\iota_\star(u) \in H^3_{\text{mot}}(S, \mathbf{Z}(2))$ (or more generally formal sums of curves and units as in Landsburg's description).

Case (1) never been carried out (and people have tried very hard to make it work without success). Case (2) leads to the construction of the *Euler system of Beilinson–Flach elements*, which will discuss in the next chapter. Here, S is the product of two modular curves, and like in Kato's construction, the basic input are Siegel units.

[3]If we start with a projective curve, then we can just delete points to make it affine, without modifying the H^1 too much – we just add copies of the trivial representation to H^1.

CHAPTER 3

An Euler system for the tensor product of modular forms

In this chapter, we'll build an explicit example of an Euler system for a 4-dimensional Galois representation V, given as the tensor product of two 2-dimensional representations arising from modular forms.

3.1. Modular curves and modular forms

(*References*: Diamond–Shurman [**DS05**], Darmon–Diamond–Taylor [**DDT97**].)

We're particularly interested in the Galois representations associated to modular forms, which come from geometry via modular curves. We'll mostly stick to weight 2 modular forms, as these are the simplest to handle; everything we'll say generalises to any weight ≥ 2 with a bit more work (but weight 1 is more difficult).

3.1a. Modular curves. For $N \geq 1$ let

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : c = 0, d = 1 \bmod N \right\}.$$

This acts on the upper half-plane $\mathcal{H}$ via $\tau \mapsto \frac{a\tau+b}{c\tau+d}$. It turns out that the quotient is naturally an algebraic variety:

Theorem 9. *For $N \geq 4$ there is an algebraic variety $Y_1(N)$ over $\mathbf{Q}$ with the following properties:*

- *$Y_1(N)$ is a smooth geometrically connected affine curve.*
- *For any field extension*[1] *$F/\mathbf{Q}$, the F-points of $Y_1(N)$ biject with isomorphism classes of pairs (E, P), where E/F is an elliptic curve and $P \in E(F)$ is a point of order N on E.*
- *$Y_1(N)(\mathbf{C}) \cong \Gamma_1(N)\backslash\mathcal{H}$, via the map sending $\tau \in \mathcal{H}$ to (E_τ, P_τ) where $E_\tau = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$ and $P_\tau = 1/N \bmod \mathbf{Z} + \mathbf{Z}\tau$.*

(Much stronger theorems are known – for instance, $Y_1(N)$ has a canonical smooth model over $\mathbf{Z}[1/N]$ – but we won't need this just now.)

Remark. There are two different choices of conventions for $\mathbf{Q}$-models for $Y_1(N)$; everyone agrees what $Y_1(N)$ means over $\mathbf{C}$, but there are two different ways to descend it to $\mathbf{Q}$, classifying elliptic curves with either a point of order N (our convention) or an embedding of the group scheme μ_N (the alternative convention). ⋄

3.1b. Galois representations. We can use these rational models of modular curves to attach Galois representations to modular forms. Let $f = \sum a_n q^n$ be a cuspidal modular eigenform of weight 2 and level $\Gamma_1(N)$, normalised so that $a_1 = 1$. By a theorem of Shimura, there is a number field L such that all $a_n \in L$. We shall fix an embedding $\iota : L \hookrightarrow \overline{\mathbf{Q}}_p$, and assume that our p-adic coefficient field $E/\mathbf{Q}_p$ contains the image of ι.

Definition. *We let $V_p(f)$ be the largest subspace of $H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes E$ on which the Hecke operators $T(\ell)$, for $\ell \nmid N$, act as multiplication by $a_\ell(f)$.*

We're mostly interested in the case when f is *new* (i.e. doesn't come from level M for any smaller $M \mid N$), in which case $V_p(f)$ is a direct summand, not just a subspace, and we have the following:

[1] Any $\mathbf{Q}$-algebra, in fact; this is important if you want to make precise the idea that $Y_1(N)$ represents a functor.

(1) $V_p(f)$ is 2-dimensional and irreducible.
(2) The L-function is given by

$$L(V_p(f), s) = L(f, s) := \sum_{n \geq 1} a_n n^{-s}.$$

When constructing Euler systems, we'll generally work with the dual $V_p(f)^*$. Since $V_p(f)$ is a subspace of $H^1(Y_1(N)_{\overline{\mathbf{Q}}}, E)$ one can show (using Poincaré duality for the compactified modular curve $X_1(N)$) that $V_p(f)^*$ is a quotient of $H^1(Y_1(N)_{\overline{\mathbf{Q}}}, E(1))$ [note the twist by 1 here].

3.2. Siegel units

(References: §§1–2 of [**Kat04**] are the definitive source.)

3.2a. The construction. Let L be any subfield of $\mathbf{C}$.

Definition. *A **modular unit** of level $\Gamma_1(N)$, defined over L, is a unit in the coordinate ring of the algebraic variety $Y_1(N)/L$.*

Since the $\mathbf{C}$-points of the modular curve $Y_1(N)$ are $\Gamma_1(N)\backslash\mathcal{H}$, we can identify

$$\left(\begin{array}{c}\text{modular units of level}\\ \Gamma_1(N)\text{ defined over }\mathbf{C}\end{array}\right) \longrightarrow \left(\begin{array}{c}\text{nowhere-zero holomorphic fcns on}\\ \Gamma_1(N)\backslash\mathcal{H}\text{ with finite-order poles at cusps}\end{array}\right).$$

In particular, a modular unit has a q-expansion in $\mathbf{C}((q))$, of the form $\sum_{n=-R}^{\infty} a_n q^n$. One might expect that the modular units defined over L are exactly the ones with $a_n \in L$ for all n, and this is almost true, but not quite: we have to mess around a bit with powers of the N-th root of unity $\zeta_N = \exp(2\pi i/N)$.

Proposition 10. *A modular unit u is defined over L if, and only if, the coefficients lie in $L(\zeta_N)$ and satisfy*

$$a_n(u)^\sigma = a_n\left(\langle \kappa_N(\sigma)\rangle \cdot u\right)$$

for every $\sigma \in \mathrm{Gal}(L(\zeta_N)/L)$, where κ_N is the mod N cyclotomic character, and $\langle d\rangle$ denotes the action of any matrix in $\mathrm{SL}_2(\mathbf{Z})$ congruent to $\left(\begin{smallmatrix} * & * \\ 0 & d\end{smallmatrix}\right) \bmod N$.

We're going to construct some "special" modular units, using nothing but classical 19th-century elliptic function theory. These functions are called **Siegel units** and they are really amazingly powerful gadgets. In fact, you can recover virtually every known example of an Euler system by starting from Siegel units!

Definition. *Let $\beta \in \mathbf{Q}/\mathbf{Z}$ be non-zero. Define the function $g_\beta : \mathcal{H} \to \mathbf{C}$ as follows:*

$$g_\beta(\tau) = e^{2\pi i\tau/12} \prod_{n \geq 0}\left(1 - e^{2\pi i(n\tau + \beta)}\right) \prod_{n \geq 1}\left(1 - e^{2\pi i(n\tau - \beta)}\right).$$

This is almost a modular unit, but not quite: if $\beta \in \frac{1}{N}\mathbf{Z}$, then acting on g_β by an element of $\Gamma_1(N)$ multiplies it by a root of unity. These error terms can be killed by a very simple modification:

Definition (Siegel units). *For $c > 1$ coprime to 6 and to the order of β, let*

$$_c g_\beta = \frac{(g_\beta)^{c^2}}{g_{c\beta}}.$$

Proposition 11. *If $\beta \in \frac{1}{N}\mathbf{Z}$, then $_c g_\beta$ is a modular unit of level $\Gamma_1(N)$, defined over $\mathbf{Q}$.* □

We can also get rid of c in a different way, by tensoring the group of modular units with $\mathbf{Q}$: this gives us a well-defined element

$$g_{1/N} \in \mathcal{O}(Y_1(N))^\times \otimes \mathbf{Q} = H^1_{\mathrm{mot}}(Y_1(N), \mathbf{Q}(1)).$$

One can check that the units $g_{1/N}$ and $_c g_{1/N}$ for varying N (coprime to c in the latter case) are compatible under pushforward morphisms along the quotient maps $Y_1(M) \to Y_1(N)$. See Kato for details.

Remark. More generally we can define a unit ${}_c g_{\alpha,\beta}$, for any $(\alpha, \beta) \in (\mathbf{Q}/\mathbf{Z})^2$, with ${}_c g_\beta = {}_c g_{0,\beta}$. However, the ${}_c g_{\alpha,\beta}$ with $\alpha \neq 0$ don't live at $\Gamma_1(N)$ level, but rather at level $\Gamma(N)$, where $\Gamma(N) \subset \mathrm{SL}_2(\mathbf{Z})$ is the "principal congruence subgroup" of matrices congruent to the identity mod N. In these lectures we'll only use the ${}_c g_\beta$. ◇

3.3. The construction

We want to study the 4-dimensional Galois representation

$$V = V_p(f)^* \otimes V_p(g)^*,$$

for f, g two weight 2 newforms. By the Künneth formula, V appears in H^2 of the affine surface $X = Y_1(N_f) \times Y_1(N_g)$. Note also that $H^3(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_p(n)) = 0$ for all n, since the étale cohomology of an affine variety vanishes in degrees above the dimension.

For $m \geq 1$, let us define $Z_m = Y_1(m^2 N)$, where N is the lowest common multiple of N_f and N_g.

Proposition 12. *There is a finite map $\iota_m : Z_m \to X$ which corresponds to $\tau \mapsto (\tau, \tau + \frac{1}{m})$ on the upper half-plane. This map is defined over $\mathbf{Q}(\mu_m)$ [but not over $\mathbf{Q}$ in general].*

Definition. *Define the class*

$${}_c\Xi_m = (\iota_m)_* \left({}_c g_{1/m^2 N}\right) \in H^3_{\mathrm{mot}}(X_{\mathbf{Q}(\mu_m)}, \mathbf{Q}(2)).$$

Define

$${}_c\mathrm{BF}^{(f,g)}_m \in H^1(\mathbf{Q}(\mu_m), V)$$

to be the image of ${}_c\Xi_m$ under the composition of maps

$$\begin{aligned} H^3_{\mathrm{mot}}(X_{\mathbf{Q}(\mu_m)}, \mathbf{Q}(2)) &\xrightarrow{r_{\text{ét}}} H^3_{\text{ét}}(X_{\mathbf{Q}(\mu_m)}, \mathbf{Q}_p(2)) \\ &\longrightarrow H^1(\mathbf{Q}(\mu_m), H^2_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_p(2)) \\ &\longrightarrow H^1(\mathbf{Q}(\mu_m), V). \end{aligned}$$

These classes all land in a lattice (the image of the étale cohomology with $\mathbf{Z}_p$ coefficients). Surprisingly they are *almost*, but not quite, an Euler system! There are polynomials $Q_\ell(t)$ such that

$$\mathrm{norm}({}_c\mathrm{BF}^{(f,g)}_{m\ell}) = Q_\ell(\sigma_\ell^{-1})\,{}_c\mathrm{BF}^{(f,g)}_m,$$

and we have

$$Q_\ell(t) \cong P_\ell(V^*, t) \pmod{\ell - 1},$$

which is sufficient: we can find an Euler system (c_m) with

$$c_m = {}_c\mathrm{BF}^{(f,g)}_m + (\text{linear combination of } {}_c\mathrm{BF}^{(f,g)}_{m'} \text{ for } m' < m),$$

so in particular $c_1 = {}_c\mathrm{BF}^{(f,g)}_1$.

Remark. By working with cohomology with coefficients in a suitable local system, we can similarly construct an Euler system for the representation $V = V_p(f)^* \otimes V_p(g)^*(-j)$, where f, g are of weights $k+2, \ell+2 \geq 2$, respectively, and $0 \leq j \leq \min\{k, \ell\}$. ◇

3.4. Siegel units and Eisenstein series

We want to focus on the following question: why should we expect the Beilinson–Flach element to have any interesting properties? The key is its relation to values of the Rankin–Selberg L-function, via Beilinson's regulator. To build up to this, we need to revisit the Siegel units, and explain why they are "motivic incarnations" of Eisenstein series.

3.4a. Relation to algebraic Eisenstein series. For any curve Y over a field k, there is a map dlog, $f \mapsto \frac{df}{f}$, which sends $\mathcal{O}(Y)^\times \otimes k$ to differential forms on Y with (at worst) simple poles along the boundary points.

If $Y = Y_1(N)$ these differentials are exactly the weight 2 modular forms of level $\Gamma_1(N)$ (while the differential forms regular at the cusps are the cusp forms).

Proposition 13. *For $\beta \in (\frac{1}{N}\mathbf{Z})/\mathbf{Z}$, the Siegel unit g_β maps to a weight 2 Eisenstein series $E^{(2)}_\beta$.*

This is easy to check: dlog maps products to sums, and g_β is defined by an infinite product, so we just apply it term-by-term to get the Fourier expansion of $E^{(2)}_\beta$, and recognise it as a linear combination of the standard basis of weight 2 Eisenstein series (see e.g. Miyake or Diamond+Shurman).

3.4b. Relation to real-analytic Eisenstein series.

Definition. *Let $\beta \in \mathbf{Q}/\mathbf{Z} - \{0\}$, and $k \in \mathbf{Z}_{\ge 0}$.*

For $s \in \mathbf{C}$ with $\Re(s) > 1 - \frac{k}{2}$, let ${}^\infty E^{(k)}_\beta(\tau, s)$ be the function on the upper half-plane $\mathcal{H}$ defined by

$$ {}^\infty E^{(k)}_\beta(\tau, s) = (-2\pi i)^{-k} \pi^{-s} \Gamma(s+k) \sum_{(c,d)\in \mathbf{Z}^2} \frac{\Im(\tau)^s}{(c\tau + d + \beta)^k |c\tau + d + \beta|^{2s}}. $$

This function is holomorphic in s for fixed τ (although not in τ, of course), and it can be extended to all $s \in \mathbf{C}$ by analytic continuation in s, with a functional equation relating s and $1 - k - s$. As a function of τ, it transforms like a modular form of weight k and level $\Gamma_1(N)$, where N is the denominator of β. (One can roughly think of it as an "L-series valued in weight k modular forms".)

If we specialising at $s = 0$ or $s = 1 - k$, we do get a holomorphic function of τ; if $k = 2$, the specialisation at $s = 1 - k$ is exactly the algebraic Eisenstein series $E^{(2)}_\beta$ above. More generally, for any $0 \ge s \ge 1 - k$, the function ${}^\infty E^{(k)}_\beta(-, s)$ is an algebraic object: a *nearly-holomorphic* modular form in the sense of Shimura (a section of a certain vector bundle over $Y_1(N)_{\mathbf{Q}}$). We call this range of integers $0 \ge s \ge 1 - k$ the *critical range.*

The following result (Kronecker's second limit formula) relates Siegel units to values of ${}^\infty E$ *outside* the critical range:

Theorem 14 (Kronecker). *We have ${}^\infty E^{(0)}_\beta(\tau, 0) = -\log|g_\beta(\tau)|$, for any $\tau \in \Gamma_1(N)\backslash\mathcal{H}$.*

3.4c. Relation to p-adic analytic Eisenstein series. There is a theory of "p-adic modular forms" due to Katz and Hida (among others), and we can build a family

$$ {}^p E^{(k)}_\beta(-, s), \qquad s \text{ a } p\text{-adic parameter} $$

of weight k p-adic Eisenstein series, for any fixed k.

If $k \ge 1$ and s is an integer with $1 - k \le s \le 0$, then we saw above that ${}^\infty E^{(k)}_\beta(-, s)$ is the image of an algebraic section of a vector bundle; and it turns out that ${}^p E^{(k)}_\beta(-, s)$ is the image of the same algebraic object in p-adic geometry. However, for other values of s, ${}^p E^{(k)}_\beta(-, s)$ is a genuinely p-adic analytic object.

Theorem 15 (p-adic Kronecker limit formula). *We have*

$$ {}^p E^{(0)}_\beta(-, 0) = -(1 - \tfrac{\varphi}{p}) \log_p(g_\beta) $$

where $\log_p$ is the p-adic logarithm, and φ the Frobenius at p.

3.4d. Interpretation in terms of cohomology. Recall that we had two "analytic" cohomology theories:

- Deligne–Beilinson cohomology, represented by pairs of an algebraic differential form and a real-analytic integral of it;
- syntomic cohomology, represented by pairs of an algebraic differential form and a p-adic integral.

Using the above formulae, we can write the image of g_β in either theory using Eisenstein series:

- the class $r_{\mathcal{B}}(g_\beta)$ is represented by $(E_\beta^{(2)}, {}^\infty E_\beta^{(0)}(-,0))$;
- the class $r_{\mathrm{syn}}(g_\beta)$ is represented by $(E_\beta^{(2)}, {}^p E_\beta^{(0)}(-,0))$.

This is the key input in relating our Euler system to values of L-functions.

3.5. The Rankin–Selberg integral formula

3.5a. The Rankin–Selberg L-function. The L-function attached to representation $V^* = V_p(f) \otimes V_p(g)$ is a rather classical object: it's the so-called *Rankin–Selberg convolution L-function* of f and g, denoted by $L(f \otimes g, s)$. This makes sense for any two newforms f, g of weights $k+2, \ell+2$, with $k, \ell \geq 0$ as before; up to finitely many bad Euler factors, this agrees with the Dirichlet series

$$L(\chi_f \chi_g, 2s-2-k-\ell) \sum_{n \geq 1} a_n(f) a_n(g) n^{-s}. \tag{‡}$$

3.5b. Integral formulae. In the 1930s, Rankin and Selberg discovered the following *integral formula* for the L-function:

THEOREM 16. *Suppose $k \geq \ell$. Then we have*

$$\int_{\Gamma_1(N)\backslash\mathcal{H}} f(-\bar\tau) g(\tau)\, {}^\infty E_{1/N}^{(k-\ell)}(\tau, s-k-1)\, \Im(\tau)^k \,\mathrm{d}\tau \wedge \mathrm{d}\bar\tau = (*) \cdot L(f \otimes g, s), \tag{1}$$

where $(*)$ is an explicit factor.

This is surprisingly simple to prove: after substituting in the sum defining the Eisenstein series, and interchanging summation and integration, you get the integral of $f(-\bar\tau)g(\tau)$ times a power of $\Im(\tau)$, over the region $\{x+iy : 0 \leq x \leq 1, 0 \leq y \leq \infty\}$. Substituting in the q-expansions of f and g and integrating term-by-term gives the result. (This way of evaluating integrals involving Eisenstein series is sometimes referred to as "unfolding".)

Although simple, this has a lot of important consequences; for instance, it follows easily from this formula and the properties of ${}^\infty E_\beta^{(k-\ell)}(\tau, s)$ that $L(f \otimes g, s)$ has meromorphic continuation to all $s \in \mathbf{C}$ (holomorphic unless $f = \bar g$), and satisfies a functional equation relating s and $k+\ell+3-s$.

3.5c. P-adic L-functions. We can write the Rankin–Selberg integral more compactly as a Petersson scalar product

$$\langle f^*, g \cdot {}^\infty E_{1/N}^{(k-\ell)}(-, s-k-1)\rangle,$$

where $f^*(\tau) = \overline{f(-\bar\tau)} = \sum \overline{a_n(f)} q^n$ is the eigenform with conjugate Fourier coefficients.

Hida has defined a version of the Petersson product for p-adic modular forms (under an additional hypothesis: f must be *ordinary*). We can then define a **p-adic L-function** by

$$L_p(f, g, s) = \left\langle f^*, g \cdot {}^p E_{1/N}^{(k-\ell)}(-, s-k-1)\right\rangle$$

where now s is a p-adic variable.

3.5d. Critical values. As we saw above, if $k > \ell$ and we take s in a certain range, then ${}^{\infty}E^{(k-\ell)}_{1/N}(\tau, s-k-1)$ and ${}^{p}E^{(k-\ell)}_{1/N}(\tau, s-k-1)$ are both the images of the same algebraic object (a nearly-holomorphic modular form). The p-adic and real-analytic Petersson products are compatible for such forms, so we conclude that the p-adic and complex L-functions **agree** (up to an explicit correction factor) for integers s with $\ell + 2 \le s \le k+1$. These are the so-called **critical values** of the L-function.

More geometrically, we can interpret the integral as a cup-product in the cohomology of coherent sheaves on the compactified modular curve $X_1(N)$ over $\mathbf{Q}$. For simplicity, take $s = k+1$; then we have :

$$\eta_f \cup \omega_g \cup \mathrm{Eis}^{(k-\ell)} = (*) \cdot L(f \otimes g, k+1),$$

where $\eta_f \in H^1(X_1(N), \omega^{2-k}(-D))$ and $\omega_g \in H^0(X_1(N), \omega^{\ell})$ are coherent cohomology classes attached to f and g, and $E^{(k-\ell)}_{1/N} \in H^0(X_1(N), \omega^{k-\ell})$ is an algebraic Eisenstein series.

3.5e. A Beilinson regulator formula. Let's now go back to the case $k = \ell = 0$, so f, g both have weight 2. Thus we don't have any critical values, but we do have Euler system classes.

Recall from Section 2.2 that as well as the étale regulator $r_{\text{ét}}$, there is also the Beilinson regulator on motivic cohomology, taking values in Hodge cohomology. Applied to $H^3_{\mathrm{mot}}(S, \mathbf{Z}(2))$ for a surface S, and composing it with the natural map from Hodge into de Rham cohomology, we can regard it as a map

$$r_{\mathcal{B}} : H^3_{\mathrm{mot}}(S, \mathbf{Z}(2)) \longrightarrow \left(\mathrm{Fil}^1 H^2_{\mathrm{dR}}(S_{\mathbf{C}})\right)^*.$$

For a class of the form $\mathfrak{z} = \iota_\star(u)$, where Z is a curve, $\iota : Z \hookrightarrow S$ is finite, and $u \in \mathcal{O}(Z)^\times$, we have an explicit formula for this map (due to Beilinson): $r_{\mathcal{B}}(\mathfrak{z})$ is the linear functional

$$\omega \mapsto \int_{Z(\mathbf{C})} \iota^\star(\omega) \log|u|. \tag{2}$$

If we apply this formula to the element $\mathfrak{z} = \Xi_1 \in H^3_{\mathrm{mot}}(Y_1(N_f) \times Y_1(N_g), \mathbf{Q}(2))$, and we take

$$\omega = (f(-\bar\tau)\,\mathrm{d}\bar\tau) \wedge (g(\tau)\,\mathrm{d}\tau)$$

for f, g of weight 2, then thanks to Theorem 14 we get

$$\begin{aligned}\langle r_{\mathcal{B}}(\Xi_1), \omega\rangle &= \left.\left(\int_{\Gamma_1(N)\backslash\mathcal{H}} f(-\bar\tau) g(\tau) E^{(0)}_{1/N}(\tau, s)\,\mathrm{d}\tau \wedge \mathrm{d}\bar\tau\right)\right|_{s=0} \\ &= (*) \cdot L'(f \otimes g, 1). \end{aligned} \tag{3}$$

(The derivative appears because the factor $(\star)$ has a pole at $s = 1$, forcing $L(f \otimes g, s)$ to vanish there.) In other words, the Beilinson regulator of the motivic class we used to define $\mathrm{BF}^{f,g}_1$, paired with a differential coming from f and g, computes a value of the L-function $L(f \otimes g, s)$! Since $L'(f \otimes g, 1)$ is never zero, this implies that Ξ_1 is non-zero (and moreover its projection to the (f, g)-eigenspace is nonzero).

This is pretty strong evidence that the Beilinson–Flach class is the right class to consider: it's the image under the étale regulator of the motivic class Ξ_1, which is a "motivic incarnation" of the Rankin–Selberg integral. Similarly, one can show that the Ξ_m are related to the twisted L-functions $L'(f \times g \times \chi, 1)$ where χ is a character mod m.

Remark. To recap: for some values of k, ℓ and integers s (the critical values) we have previously given a algebro-geometric interpretation of the Rankin–Selberg integral via coherent cohomology; and just now, for $k = \ell = 0$ and $s = 1$ (which is *not* a critical value), we've given it a different geometric interpretation, via Deligne–Beilinson cohomology. Using cohomology with non-constant coefficients, this extends to any k, ℓ, s with $1 \le s \le \min(k, \ell) + 1$; in particular, this range is always disjoint from the critical range.

Sadly, there are other values of $L(f \times g, s)$ which we don't know how to interpret geometrically, although the Bloch–Kato conjecture predicts it should be possible. (I would love to know how to relate the leading term of $L(f, g, s)$ at $s = 0$ to anything in algebraic geometry; this is related to the unsuccessful approach (1) in Section 2.3b.) ◇

3.6. Other Rankin–Selberg-type formulae

[This section was not presented in the live lectures, due to lack of time]

The Rankin–Selberg integral is only the first of a very wide class of formulae, which express the L-values of an automorphic form for some reductive group G in terms of its integral against an Eisenstein series on some subgroup H (a "period integral"). There is a survey article by Bump [**Bum05**]) which catalogues dozens of constructions of this kind.

So we can play the following game: if we want to build an Euler system for some class of automorphic Galois representations, then we can look for known formulae expressing the L-function of our representation in terms of periods of automorphic forms. Then we can stare at the resulting integrals and try to recognise them as Beilinson regulators of motivic cohomology classes. If we can do this, then the étale versions of these classes should be non-zero (although we can't prove this), and they are clearly the right building blocks for an Euler system for our representation.

Remark. This won't always work, sadly. Firstly, in many of the known Rankin–Selberg formulae the groups G and H do not have Shimura varieties, so they lie outside the world of algebraic geometry; there is a perfectly good Rankin–Selberg integral for $\mathrm{GL}_m \times \mathrm{GL}_n$ for any integers (m, n), but it doesn't correspond to anything motivic unless $(m, n) = (2, 2)$.

Secondly, even if G corresponds to a Shimura variety (and H to a Shimura subvariety), then there is a second stumbling block: the Eisenstein series. For most Rankin–Selberg formulae, these will not be Eisenstein series for GL_2, but for other, more general reductive groups; and we need a way to relate these to motivic cohomology, generalising the way that GL_2 Eisenstein series are related to units via Kronecker's limit formula. This seems to be a difficult problem in general. $\diamond$

Despite these apparently gloomy remarks, all is not lost: there are surprisingly many Rankin–Selberg formulae in which only GL_2 Eisenstein series appear! There's now an ongoing project, being pursued by several research groups, to build Euler systems for each such integral formula. Some examples are

- an Euler system for the Asai representation attached to quadratic Hilbert modular forms [**LLZ18**], [**Gro20**], with $H = \mathrm{GL}_2$ and $G = \mathrm{Res}^F_{\mathbf{Q}} \mathrm{GL}_2$, where F is a real quadratic field;
- an Euler system for the spin representations attached to genus 2 Siegel modular forms [**LSZ17**], with $H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$ and $G = \mathrm{GSp}_4$;
- an Euler system for Picard modular forms [**LSZ21**], with $H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{Res}^K_{\mathbf{Q}} \mathrm{GL}_1$ and $G = \mathrm{GU}(2, 1)$, where K is an imaginary quadratic field and $\mathrm{GU}(2, 1)$ a unitary group split over K. In this case, we get an *Euler system over* K: in other words, we construct cohomology classes over all the finite abelian extensions of K.

CHAPTER 4

Evaluating syntomic regulators

4.1. Fontaine's theory

We will summarize here briefly some results from p-adic Hodge theory that we will need in what follows. For a more detailed account of this beautiful theory, see [**Ber05**].

4.1a. Period rings. Let V be a p-adic representation of $G_{\mathbf{Q}_p}$ of dimension d, and we write $\mathbf{B}_{\text{cris}} \subset \mathbf{B}_{\text{dR}}$ for Fontaine's rings of periods. We can then attach to V invariants

$$\mathbf{D}_{\text{cris}}(V) = (V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{cris}})^{G_{\mathbf{Q}_p}}, \qquad \mathbf{D}_{\text{dR}}(V) = (V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{dR}})^{G_{\mathbf{Q}_p}}$$

which are finite-dimensional $\mathbf{Q}_p$-vector space, of dimension $\leq d$, equipped with a decreasing filtration $\operatorname{Fil}^\bullet$ and - in the case of $\mathbf{D}_{\text{cris}}(V)$ - with a Frobenius operator φ. The filtration has the property that $\operatorname{Fil}^0 \mathbf{D}_{\text{dR}}(V(n)) = \operatorname{Fil}^n \mathbf{D}_{\text{dR}}(V)$.

Definition. *We say that V is* crystalline *if* $\dim_{\mathbf{Q}_p} \mathbf{D}_{\text{cris}}(V) = d$ *and* de Rham *if* $\dim_{\mathbf{Q}_p} \mathbf{D}_{\text{dR}}(V) = d$.

Remark. In particular, any crystalline representation is automatically de Rham. In this case, we have $\mathbf{D}_{\text{cris}}(V) = \mathbf{D}_{\text{dR}}(V)$, and we get an action of φ on $\mathbf{D}_{\text{dR}}(V)$ by transport of structure: it is a "filtered φ-module". ◇

We have $\mathbf{D}_{\text{dR}}(\mathbf{Q}_p(1)) \cong \mathbf{Q}_p$, so we get a pairing

$$\langle \quad , \quad \rangle_{\text{dR}} : \mathbf{D}_{\text{dR}}(V) \times \mathbf{D}_{\text{dR}}(V^*(1)) \longrightarrow \mathbf{Q}_p, \tag{4}$$

which is perfect if V (and hence $V^*(1)$)) are de Rham.

4.1b. Comparison isomorphisms. The "purpose" of the period rings is to relate étale cohomology with other theories:

Theorem 17 (Faltings, Tsuji). *If $X/\mathbf{Q}_p$ is a smooth variety, then there are canonical isomorphisms*

$$\mathbf{D}_{\text{dR}}(H^i_{\text{ét}}(X, \mathbf{Q}_p)) \cong H^i_{\text{dR}}(X)$$

compatible with filtrations.

If furthermore X has good reduction, we have another canonical isomorphism

$$\mathbf{D}_{\text{cris}}(H^i_{\text{ét}}(X, \mathbf{Q}_p)) \cong H^i_{\text{cris}}(X_0)$$

compatible with φ, where H^i_{cris} denotes crystalline cohomology and $X_0/\mathbf{F}_p$ is the special fibre of X.

4.1c. Exponential maps. A remarkable theorem of Bloch–Kato [**BK90**] shows that the period rings fit into an exact sequence

$$0 \to \mathbf{Q}_p \to \mathbf{B}_{\text{cris}} \to \mathbf{B}_{\text{cris}} \oplus \mathbf{B}_{\text{dR}} / \operatorname{Fil}^0 \to 0 \tag{5}$$

where the middle map is $x \mapsto ((1-\varphi)x, x \bmod \operatorname{Fil}^0)$. Tensoring with V and taking cohomology, we get

$$0 \to V^{G_{\mathbf{Q}_p}} \to \mathbf{D}_{\text{cris}}(V) \to \mathbf{D}_{\text{cris}}(V) \oplus \mathbf{D}_{\text{dR}}(V) / \operatorname{Fil}^0 \xrightarrow{\exp_{\text{BK}}} H^1(\mathbf{Q}_p, V) \to \dots$$

The image of $\exp_{\text{BK}}$ is exactly the Bloch–Kato subspace $H^1_{\text{f}}(\mathbf{Q}_p, V)$ which we saw in Chapter 1.

If V is crystalline, then restricting $\exp_{\text{BK}}$ to $\mathbf{D}_{\text{cris}}$ gives an isomorphism

$$\exp_{\text{BK}} : \mathbf{D}_{\text{dR}}(V) / (1-\varphi) \operatorname{Fil}^0 \mathbf{D}_{\text{dR}}(V) \xrightarrow{\cong} H^1_{\text{f}}(\mathbf{Q}_p, V).$$

4.2. Syntomic cohomology

The idea of syntomic cohomology is to "build a cohomology theory around" the fundamental sequence (5).

4.2a. Syntomic cohomology. Let X be a smooth $\mathbf{Z}_p$-scheme with special fibre X_0 and generic fibre $X_{\mathbf{Q}_p}$. We can the consider de Rham cohomology $R\Gamma_{\mathrm{dR}}(X_K)$, equipped with a decreasing filtration $\mathrm{Fil}^\bullet$, and rigid cohomology $R\Gamma_{\mathrm{rig}}(X_0)$ equipped with a Frobenius operator φ. (Rigid cohomology is isomorphic to crystalline cohomology, but easier to write down.)

We have a *specialisation map*

$$\mathrm{sp} : R\Gamma_{\mathrm{dR}}(X_K) \longrightarrow R\Gamma_{\mathrm{rig}}(X_0),$$

which is a quasi-isomorphism if X is nice enough.

DEFINITION. *Let $n \in \mathbf{Z}$. Define syntomic cohomology $R\Gamma_{\mathrm{syn}}(X, n)$ to be the mapping fibre*

$$R\Gamma_{\mathrm{syn}}(X, n) = \mathrm{MF}\left[\mathrm{Fil}^n R\Gamma_{\mathrm{dR}}(X_K) \xrightarrow{\left(1-\frac{\varphi}{p^n}\right)\circ \mathrm{sp}} R\Gamma_{\mathrm{rig}}(X_0)\right]. \qquad (\dagger)$$

We can now state, carefully, the comparison between motivic, étale, and syntomic cohomology:

THEOREM 18 (Nekovar–Niziol). *For any i, n there is a commutative diagram of maps*

$$\begin{array}{ccc} H^i_{\mathrm{mot}}(X, \mathbf{Q}(n)) & \longrightarrow & H^i_{\mathrm{mot}}(X_{\mathbf{Q}_p}, \mathbf{Q}(n)) \\ \downarrow r_{\mathrm{syn}} & & \downarrow r_{\text{ét}} \\ H^i_{\mathrm{syn}}(X, n) & \xrightarrow{\mathrm{comp}} & H^i_{\text{ét}}(X_{\mathbf{Q}_p}, \mathbf{Q}_p(n)) \end{array}$$

Moreover, there is a morphism of spectral sequences, from the spectral sequence of the mapping fibre converging to $H^i_{\mathrm{syn}}(X, n)$, to the Hochschild–Serre sequence converging to $H^i_{\text{ét}}$. So if $\mathrm{Fil}^1 H^i_{\mathrm{syn}}$ *denotes the kernel of $H^i_{\mathrm{syn}}(X, n) \to \mathrm{Fil}^n H^i_{\mathrm{dR}}(X_{\mathbf{Q}_p})$, then we have a second diagram*

$$\begin{array}{ccc} \mathrm{Fil}^1 H^i_{\mathrm{syn}}(X, n) & \xrightarrow{\mathrm{comp}} & \mathrm{Fil}^1 H^i_{\text{ét}}(X_{\mathbf{Q}_p}, n) \\ \downarrow & & \downarrow \\ H^{i-1}_{\mathrm{dR}}/(1 - p^{-n}\varphi)\,\mathrm{Fil}^n & \xrightarrow[\exp_{\mathrm{BK}}]{} & H^1(\mathbf{Q}_p, H^{i-1}_{\text{ét}}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)(n)) \end{array}$$

4.2b. Finite-polynomial cohomology. From the mapping fibre ($\dagger$), we see that: if $\omega \in \mathrm{Fil}^n H^i_{\mathrm{dR}}(X_K)$, and $\mathrm{sp}(\omega)$ lands in the $\varphi = p^n$ eigenspace of $H^i_{\mathrm{rig}}(X_0)$, then ω lifts to some $\tilde{\omega} \in H^i_{\mathrm{syn}}(X, n)$ (in general non-uniquely).

In *op.cit.*, Besser defines *finite polynomial cohomology* by replacing $1 - \varphi/p^n$ with more general polynomials $P(p^{-n}\varphi)$, where $P \in 1 + t\mathbf{Q}_p[t]$. If we are given any class in $\omega \in \mathrm{Fil}^n H^i_{\mathrm{dR}}(X_K)$, we can always choose P so that $P(p^{-n}\varphi)$ kills $\mathrm{sp}(\omega)$, so ω lifts to fp-cohomology. We can think of this lifting as a generalisation of Coleman integration; this is the point of view taken by Besser [**Bes12**].

4.2c. Duality. We also have a version with compact support, $R\Gamma_{\mathrm{fp},c}(X, n, P)$, which is defined similarly; and there are cup-products

$$H^i_{\mathrm{syn}}(X, n) \times H^j_{\mathrm{fp,c}}(X, m, P) \longrightarrow H^{i+j}_{\mathrm{fp},c}(X, m + n, P). \qquad (6)$$

If the polynomial $P(t)$ satisfies $P(p^{-1}) \neq 0$, and if X is connected of dimension d, then there is a canonical isomorphism

$$\mathrm{tr}_{\mathrm{fp},X} : H^{2d+1}_{\mathrm{fp},c}(X, d + 1; P) \xrightarrow{\cong} \mathbf{Q}_p.$$

We hence obtain a pairing

$$\langle -, -\rangle_{\mathrm{fp},X} : H^i_{\mathrm{syn}}(X, n) \times H^{2d+1-i}_{\mathrm{fp,c}}(X, d + 1 - n, P) \longrightarrow \mathbf{Q}_p. \qquad (7)$$

Pushforward and pullback along closed immersions are adjoint with respect to this pairing. In particular, if we consider a closed embedding $\iota : Z \hookrightarrow S$ of a curve into a surface, and we take $x \in H^1_{\mathrm{syn}}(Z, 1)$ and $y \in H^2_{\mathrm{fp},c}(S, 1; P)$, then

$$\langle \iota_*(x),\, y\rangle_{\mathrm{fp},X} = \langle x,\, \iota^*(y)\rangle_{\mathrm{fp},Z}.$$

4.3. Strategy

We now go back to the situation in the previous chapter: let $V = V_p(f)^* \otimes V_p(g)^*$, where f, g are cuspidal new eigenforms of weight 2 and level N. We'll write Y for $Y_1(N)$ here.

We assume that both modular forms are *p-ordinary*, so their p-th Fourier coefficients $a_{f,p}$, $a_{g,p}$ are p-adic units. (Actually we only need this for f.) One can show that

- the representation $V|_{G_{\mathbf{Q}_p}}$ is crystalline;
- one can identify $\mathbf{D}_{\mathrm{dR}}(V)$ with a direct summand of $H^2_{\mathrm{dR}}(Y^2)$;
- the operator $(1 - \varphi)$ is invertible on $\mathbf{D}_{\mathrm{dR}}(V)$.

By (†) we obtain a map

$$H^3_{\mathrm{syn}}(Y^2, 2) \longrightarrow \mathbf{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V)$$

which fits into the commutative diagram

$$\begin{array}{ccccc} & & H^3_{\mathrm{mot}}(Y^2, \mathbf{Q}(2)) & & \\ & \swarrow^{r_{\mathrm{syn}}} & & \searrow^{r_{\mathrm{\acute{e}t}}} & \\ H^3_{\mathrm{syn}}(Y^2, 2) & & & & H^3_{\mathrm{\acute{e}t}}(Y^2, \mathbf{Q}_p(2)) \\ \downarrow{\scriptstyle \mathrm{pr}^{(f,g)}} & & & & \downarrow{\scriptstyle \mathrm{pr}^{(f,g)}} \\ \dfrac{\mathbf{D}_{\mathrm{dR}}(V)}{\operatorname{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V)} & & \xrightarrow{\quad \exp_{\mathrm{BK}} \quad} & & H^1_{\mathrm{f}}(\mathbf{Q}_p, V) \end{array}$$

Now under the pairing (4), we have an identification

$$\frac{\mathbf{D}_{\mathrm{dR}}(V)}{\operatorname{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V)} \cong \left(\operatorname{Fil}^1 \mathbf{D}_{\mathrm{dR}}(V^*)\right)^*.$$

Idea. Evaluate the pairing

$$\left\langle \mathrm{pr}^{(f,g)} \circ r_{\mathrm{syn}} \circ \iota_*(g_{1/N}),\, \lambda \right\rangle_{\mathrm{dR}} \tag{8}$$

for a suitable $\lambda \in \operatorname{Fil}^1 \mathbf{D}_{\mathrm{dR}}(V^*)$. We will take $\lambda = \eta \otimes \omega$, for carefully chosen elements

$$\eta \in \operatorname{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V_p(f)) \quad \text{and} \quad \omega \in \operatorname{Fil}^1 \mathbf{D}_{\mathrm{dR}}(V_p(g)).$$

These elements have the following properties:

(1) They are p-adic analogues of the differentials

$$f(-\bar{\tau})d\bar{\tau} \quad \text{and} \quad g(\tau)d\tau$$

which we saw in the Rankin–Selberg integral formula.

(2) For $? \in \{f, g\}$, write $\alpha_?$, $\beta_?$ for the roots of the Hecke polynomial

$$t^2 - a_p(?)t + p\varepsilon_?(p),$$

and suppose α_f denotes the unit root. Then we arrange that η_f is in the $\varphi = \alpha_f$ eigenspace.

It follows that λ is annihilated by $R(\frac{\varphi}{p})$, where

$$R(t) = (1 - \tfrac{pt}{\alpha_f \alpha_g})(1 - \tfrac{pt}{\alpha_f \beta_g}).$$

Remark. The roles of f and g are *not* symmetric! ⋄

Step 1: Lift to fp-cohomology

Thanks to (†), there are distinguished lifts $\tilde{\eta}$ and $\tilde{\omega}$ to fp-cohomology with compact support. Then

$$\tilde{\lambda} = \tilde{\eta} \otimes \tilde{\omega} \in H^2_{\mathrm{fp},c}(Y^2, 1; R),$$

and we have

$$\begin{aligned}
\left\langle \mathrm{pr}^{(f,g)} \circ r_{\mathrm{syn}} \circ \iota_*(g_{1/N}),\, \lambda \right\rangle_{\mathrm{dR}} &= \left\langle r_{\mathrm{syn}} \circ \iota_*(g_{1/N}), \tilde{\lambda} \right\rangle_{\mathrm{fp}, Y\times Y} \\
&= \left\langle \iota_* \left(r_{\mathrm{syn}}(g_{1/N})\right), \tilde{\lambda} \right\rangle_{\mathrm{fp}, Y\times Y} \\
&= \left\langle r_{\mathrm{syn}}(g_{1/N}), \iota^*(\tilde{\lambda}) \right\rangle_{\mathrm{fp}, Y} \quad \text{(push-pull adjunction)} \\
&= \mathrm{tr}_{\mathrm{fp},Y} \left(r_{\mathrm{syn}}(g_{1/N}) \cup \tilde{\eta} \cup \tilde{\omega}\right). \qquad (9)
\end{aligned}$$

Step 2: An explicit formula

As we saw in the last lecture, $r_{\mathrm{syn}}(g_{1/N})$ is described explicitly via a p-adic analogue of Kronecker's limit formula:

Proposition 19. *$r_{\mathrm{syn}}(g_{1/N})$ is given by the pair of Eisenstein series*

$$(E^{(2)}_{1/N}, {}^{p}E^{(0)}_{1/N}(-,0)).$$

We have a similar description of $\tilde{\eta}_c$ and $\tilde{\omega}$ as pairs of global sections, which involve the differentials η_f and ω_g (c.f. Section 3.5d) and p-adic integrals of these. By using Besser's explicit formulae for the cup product pairing for fp-cohomology, we finally deduce that (9) is equal to

$$\frac{1}{R(1)} \langle f^*, g \cdot {}^{p}E^{(0)}_{1/N}(-,0)\rangle.$$

where the bracket is Hida's Petersson product. By construction, this is the p-adic L-value $L_p(f, g, 1)$, giving a p-adic analogue of (3).

4.3a. Twisting. We have only dealt with weight 2 forms above, but a similar argument shows that if f, g have any weights $k+2, \ell+2 \geq 2$, then our geometrically-defined cohomology class for $[V_p(f) \otimes V_p(g)]^*(-j)$ is related to the p-adic L-value $L_p(f, g, 1+j)$, for all $0 \leq j \leq \min(k, \ell)$.

By p-adic analytic continuation (not just in j but also in k, ℓ), we can pass from this "geometric" range $0 \leq j \leq \min(k, \ell)$ to the "critical" range where $k > \ell$ and $\ell + 1 \leq j \leq k$. The analytically-continued statement relates the Soulé twists of our Euler system, landing in the cohomology of $[V_p(f) \otimes V_p(g)]^*(-j)$ for these j, to the critical L-values $L(f, g, 1+j)$.

This relation, together with the general Euler system machine for bounding Selmer groups, gives the proof of the Bloch–Kato conjecture for all non-zero critical values of Rankin–Selberg L-functions.

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