Bernstein-Zelevinsky classification (II)

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Goal Understand how parabolically induced reps decompose.

Notation
- $G_n = GL_n(F)$ F local field
- $\alpha = (n_1, \dots, n_r)$ partition of n ($n_i \in \mathbb{N}$, $n_1 + \dots + n_r = n$, order matters)
- $G_\alpha = \prod G_{n_i}$
- $P_\alpha = G_\alpha \ltimes U_\alpha$ parabolic in G_n

Recall π irred rep^n of G_n

$\exists$ * partition $\alpha = (n_1 \cdots n_r)$ of n

* SC rep^n $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r$ of G_α

st π is a subrep of $\mathrm{Ind}_{P_\alpha}^{G_n}(\sigma_1 \otimes \cdots \otimes \sigma_r) =: \sigma_1 \times \cdots \times \sigma_r$

($\mathrm{Hom}($ ~~~ $) \neq 0 \Rightarrow \mathrm{Hom}(\pi, \mathrm{Ind}(\sigma_1 \otimes \cdots \otimes \sigma_r)) \neq 0$)

1) Jacquet modules of induced reps (cf Rodier §2.3)

$n = \underbrace{n_1 + \dots + n_r}_{(\alpha)}$ $\quad \rho_1 \times \cdots \times \rho_r \quad \cdots$ sc reps of n_i

$ = \underbrace{m_1 + \dots + m_s}_{(\beta)}$, $\quad \sigma_1 \quad \cdots \times \sigma_s \quad \cdots$ " m_j

What is $\mathrm{Hom}_{G_n}(\rho_1 \times \cdots \times \rho_r, \sigma_1 \times \cdots \times \sigma_s)$?

$= \mathrm{Hom}_{G_\beta}(r_{U_\beta}(\rho_1 \times \cdots \times \rho_r), \sigma_1 \otimes \cdots \otimes \sigma_s)$,

so want to compute $r_{U_\beta}(\rho_1 \times \cdots \times \rho_r)$.

Notation For two partitions α, β of n, say α refines

β if $\alpha = n_1 \cdots n_r$
$\beta = m_1 \cdots m_s$ & can write

$\left.\begin{array}{l} n_1 + \dots + n_{i_1} = m_1 \\ n_{i_1+1} + \dots + n_{i_2} = m_2 \end{array}\right\}$ etc, some i_j so $P_\alpha \subset P_\beta$.

e.g. $(1,1,2)$ refines $(2,2)$

Prop $r_{U_\beta}(\rho_1 \times \cdots \times \rho_r)$ has a filtration w graded pieces

$$\mathrm{Ind}^{G_\beta}_{\underbrace{P_{z(\alpha)} \cap G_\beta}_{\text{parabolic in } G_\beta}} \left(\rho_{z(1)} \times \cdots \times \rho_{z(r)} \right)$$

z ~~sum~~ varies over permutations st $z(\alpha)$ refines β

~~where varies over st refines~~,
up to action of $S_{m_1} \times \cdots \times S_{m_s}$

Sketch of pf consider double coset space $P_\alpha \backslash G_n / P_\beta = W_{\alpha\beta}$ finite, + gives P_β-orbits on $P_\alpha \backslash G_n$.

(Mackey theory)

hence $\mathrm{Ind}^{G}_{P_\alpha}(\rho)$ has P_β-stable filt^n indexed by $w \in W_{\alpha\beta}$, gradeds

$$I(w) = c\mathrm{Ind}^{P_\beta}_{w^{-1}P_\alpha w \cap P_\beta}(\rho^w).$$

& can compute $r_{M_\beta}(c\mathrm{Ind}(\dots)) = \mathrm{Ind}^{G_\beta}_{P_\alpha^w \cap G_\beta}\left(r_{G_\alpha^w \cap P_\beta}(\rho^w) \right)$

which is 0 unless $G_\alpha^w \cap U_\beta = \{1\}$ (as ρ^w is sc)

$\Rightarrow$ α^w refine β. $\square$

Example if $\alpha = (1, \dots, 1)$ then
$[r(\rho_1 \times \cdots \times \rho_n)]^{ss} =$ all the orderings of $\rho_1, \dots \rho_n)$.

Corollary For any irred subquot ω of $\rho_1 \times \cdots \times \rho_r$, $\exists$ perm^n z of $(1, \dots, r)$ st ω is sub of $\rho_{z(1)} \times \cdots \times \rho_{z(r)}$.

Pf Follows from last prop & Frobenius recip.

Prop let $\rho_1 \times \cdots \times \rho_r$, $\sigma_1 \times \cdots \times \sigma_s$ G_n-reps, ρ_i, σ_j, sc.
Then TFAE:
(i) $\rho_1 \times \cdots \times \rho_r$, $\sigma_1 \times \cdots \times \sigma_s$ have an irred subquot in common
(ii) $r = s$ & the ρ_i are a perm^n of the σ_j.

& if this holds, then:

* $\rho_1 \times \cdots \times \rho_r$ & $\sigma_1 \times \cdots \times \sigma_s$ have same semisimplification
* $\mathrm{Hom}(\rho_1 \times \cdots \times \rho_r,\ \sigma_1 \times \cdots \times \sigma_s)$ is 1-dim'.

This is as far as one can get using only Jacquet Functor
+ generalizes to all reductive gps.

Twisted Jacquet modules

$\theta : F \to \mathbb{C}^\times$ nontriv' additive character

regard θ as char of upper-triangular mats. by

$$\begin{pmatrix} 1 & x_1 & \cdots & * \\ & \ddots & \ddots & \\ & & & x_{n-1} \\ 0 & & & 1 \end{pmatrix} \to \theta(x_1 + \cdots + x_{n-1}). \qquad (\dagger)$$

$$P_n = \left(\begin{array}{ccc|c} & & & * \\ & * & & \vdots \\ & & & * \\ \hline 0 & \cdots & 0 & 1 \end{array} \right)$$

π P_n-rep^n

U_α for $\alpha = (n-i, 1, \ldots, 1)$

$$\text{Let} \quad V_n^i = \begin{array}{cc} \overbrace{}^{n-i} & \overbrace{}^{i} \end{array}$$
$$\left(\begin{array}{c|c} \mathrm{id} & * \\ \hline 0 & \begin{matrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{matrix} \end{array} \right) \subset P_n$$

$\theta_n^i : V_n^i \to \mathbb{C}^\times$ using $(\dagger)$ on bottom right block.

(θ_n^1 = triv' char'.

Def^n $\pi^{(i)} :=$ max' quot of π on which $V_n^i = \theta_n^i$.

(rep^n of G_{n-i}). (i^{th} derivative of π.)

$\pi^{(0)} := \pi$.

(note : π = dual of space of Whittaker func's on π.)

Prop : $\exists$ filt^n $\pi = \pi_0 \supset \pi_1 \supset \cdots \supset \pi_n = 0$

$$\pi_{i-1}/\pi_i = c\mathrm{Ind}_{G_{n-i} \ltimes V_n^i}^{P_n}\left(\pi^{(i)} \otimes \theta_n^i\right)$$

Pf Uses exactness of Jacquet modules (twisted + untw.).

Thm If σ SC rep^n of G_n, then

* $\sigma^{(i)} = 0$, $1 < i \leq n-1$

* $\sigma^{(n)}$ 1-dim^l (σ is generic)

Pf $\theta_n^{(i)}$ is triv^l on $\begin{pmatrix} id_{n-i} & * \\ 0 & id_i \end{pmatrix}$, so $\pi \to \pi^{(i)}$ U_α $\alpha = (n-i, i)$

factors thru (untwisted) Jacquet functor ; $\Rightarrow$ 0 for SC rep^{ns}.
r_{U_α}

hard part is existence + uniqueness of Wh. model for SC reps (due to Gelfand + Kazhdan). $\square$

Now compute derivatives of induced reps.

Def^n R_n = Grothendieck gp of finite-length G_n-reps
= free ab gp on iso classes of irreps.

$R = \bigoplus_{n \geq 0} R_n$ – ring under parabolic induction. $(R_0 = \mathbb{Z})$
(associativity = transitivity of induction)

Why ring? $\rho_1 \in G_{n_1}$ $\rho_2 \in G_{n_2}$

$$[\rho_1] \times [\rho_2] = [\underbrace{\rho_1 \times \rho_2}_{G_{n_1+n_2}\text{ rep}^n}]$$

Fact : this is commutative – $(\rho_1 \times \cdots \times \rho_r)^{ss}$ indep of ordering.

Def^n $D(\pi) = \sum_{i=0}^{n} [\pi^{(n-i)}] \in R$ π irrep of G_n.

Prop This extends to a ring hom $R \to R$
sending $[\sigma] \mapsto 1 + [\sigma]$ for $[\sigma]$ sc.

eg $D(\rho_1 \times \rho_2) = 1 + [\rho_1] + [\rho_2] + [\rho_1 \times \rho_2]$
ρ_1, ρ_2 sc.

Reducibility theorem

def^n for π irred G_n-rep, let $\mathrm{supp}(\pi)$: unique multiset $\{\rho_1, \dots, \rho_r\}$ of SC reps st
π composition factor of $\rho_1 \times \dots \times \rho_r$.
(for $n = 0$, $\mathrm{supp}(\mathbb{1}) = \emptyset$)

Lemma: Let π G_n-rep^n
w irred P_n-subrep of π
$\sigma = w^{(k)}$ unique nonvanishing derivative of w
(rep of G_{n-k})
tensor each factor by $|\det|$
then $(\mathrm{supp}\,\sigma)(1) \subset \mathrm{supp}(\pi)$. $\quad$ ($\mathrm{supp}\,\sigma \subset \mathrm{supp}(\pi)$)

Pf Fiddly Frobenius recip argument, using duality.

Corollary If $\pi := \rho_1 \times \dots \times \rho_r$ is reducible,
then $\exists\ i \neq j$ st $\rho_i = \rho_j(1)$.

Pf WMA π has subrep τ st $\tau^{(n)} = 0$.
w irred subrep of $\tau|_{P_n}$ $\Rightarrow$ $w^{(m)} \neq 0$ some $m < n$
$\sigma = w^{(m)}$ subrep of $\tau^{(m)}$
$\Rightarrow$ $\mathrm{Supp}(\sigma)(1) \subset \{\rho_1 \cdots \rho_r\}$
$\mathrm{supp}(\sigma) \subset \{\rho_1 \cdots \rho_r\}$
$\mathrm{Supp}(\sigma) \neq \emptyset$. $\square$

(converse also true – classical argument w. intertwining operators.)

Segments $\Delta = (\rho, \rho(1), \ldots, \rho(r-1))$ ρ SC rep. of G_ℓ
$n = \ell \times r$

$Z(\Delta)$
(Rodier)

$\langle \Delta \rangle$ = unique irred subrep of $\rho \times \cdots \times \rho(r-1)$
$r_{U_\alpha}(\langle \Delta \rangle) = \rho \otimes \cdots \otimes \rho(r-1)$ [this order],
$\alpha = (\ell, \ldots, \ell)$

Thm $D(\langle \Delta \rangle) = \langle \Delta \rangle + \langle \Delta^- \rangle$,

$$\Delta^- = (\rho, \ldots, \rho(r-2)).$$

In particular $\langle \Delta \rangle$ is generic iff $r = 1$.
& it is irreducible as P_n-rep. (includes supercuspidals!)

E.g. for $n=2$, $\Delta = (1 \cdot |\,|^{-\frac{1}{2}}, 1 \cdot |\,|^{\frac{1}{2}})$
$\langle \Delta \rangle$ = triv' rep^n. of G_2

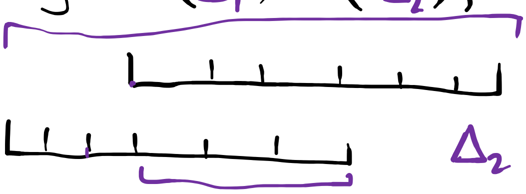
Levels $\lambda(\pi) = \max^l k$ st $\pi^{(k)} \neq 0$

Say π is homogenous if all subquots ω
of π have $\lambda(\omega) = \lambda(\pi)$. ($\Rightarrow$ composition factors of π as G_n rep all show up in $\pi^{(\lambda(\pi))}$)

Prop If $\Delta_1, \ldots, \Delta_r$ segments, Δ_i and Δ_j not linked for $i \neq j$, then $\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_r \rangle$ irred.

Pf first show $\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_r \rangle$ homogenous if
no two of Δ_i juxtaposed
secondly: top deriv is $\langle \Delta_1^- \rangle \times \cdots \times \langle \Delta_r^- \rangle$
– "less linked" than $\Delta_1 \ldots \Delta_r$, so induction on n. $\square$

in linked case: can analyse composition series
eg $\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle$,



Δ_1 Δ_2 precedes Δ_1.

Δ_2

$\Delta^\cup = \Delta_1 \cup \Delta_2$ $\Delta^\cap = \Delta_1 \cap \Delta_2$ $\omega = \langle \Delta^\cup \rangle \times \langle \Delta^\cap \rangle$

is irred

$\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle$
has ω as quot.

def $\langle \Delta_1, \Delta_2 \rangle := \ker\left(\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \twoheadrightarrow \langle \Delta^{\cup} \rangle \times \langle \Delta^{\cap} \rangle \right)$
irred repⁿ.

more generally $\langle \Delta_1, \dots, \Delta_r \rangle$ = unique irred subrep of $\langle \Delta_1 \rangle \times \dots \times \langle \Delta_r \rangle$. Δ_i doesn't precede Δ_j $i<j$.

- only depends on multiset $\{\Delta_1 \cdots \Delta_r\}$ & any multisegment has at least one permitted ordering.

Fact $\langle \Delta_1', \dots, \Delta_t' \rangle$ is a composition factor of $\langle \Delta_1 \rangle \times \dots \times \langle \Delta_r \rangle$ iff $(\Delta_1', \dots \Delta_t')$ gotten from $(\Delta_1 \dots \Delta_r)$ by finitely many iterations of replacing a linked pair (Δ_i, Δ_j) with $(\Delta^{\cup}, \Delta^{\cap})$.

e.g. support $(\rho, \rho(1), \rho(1), \rho(2))$

5 irreps : $\langle [\rho(1)], [\rho, \rho(1), \rho(2)] \rangle$

$\langle [\rho(1), \rho(2)], [\rho, \rho(1)] \rangle$

$\langle [\rho(1)], [\rho(1), \rho(2)], \rho \rangle$

$\langle \rho[2], [\rho(1)], [\rho, \rho(1)] \rangle$

generic $\rightarrow \langle [\rho(2)], [\rho(1)], [\rho(1)], [\rho] \rangle$

note: decompⁿ of induced repⁿs requires no information about supercuspidals.