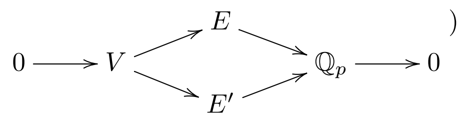
Galois Cohomology (Study Group)

An interlude (Chris Williams)

Proposition 0.1. *Let $K/\mathbb{Q}_p$ finite extension, V a finite dimensional $\mathbb{Q}_p$-vector space with a continuous G_k-action. There there is a bijection $H^1(K,V) \leftrightarrow$isomorphism classes of extensions of the trivial representations by V (i.e., short exact sequences $0 \to V \to E \to \mathbb{Q}_p \to 0$, where two sequences are isomorphism if there exists $E \cong E'$ such that*



Proof. The map is given in the following way: Given $0 \to V \to E \to \mathbb{Q}_p \to 0$, take the Galois cohomology $0 \to H^0(K,V) \to H^0(K,E) \to \mathbb{Q}_p \xrightarrow{\delta} H^1(K,V) \to \dots$. In particular, $\phi := \delta(1) \in H^1(K,V)$.

Explicitly: take $e \in E$ mapping to $1 \in \mathbb{Q}_p$, then for $g \in G_k$, $ge - e \mapsto 0 \in \mathbb{Q}_p$, there exists $v_g \in V$ with $v_g \mapsto ge - e$. Then ϕ is represented by the cocycle $g \mapsto v_g$. To define an inverse: let $\phi \in H^1(K,V)$ be represented by $\psi : G_k \to V$. Define $E := V \oplus \mathbb{Q}_p$ and give it a G_k-action by

$$\rho_E(g) = \left(\begin{array}{c|c} \rho_v(g) & \psi(g) \\ \hline 0 & 1 \end{array}\right)$$

$$\rho_E(g)\rho_E(h) = \left(\begin{array}{c|c} \rho_v(g)\rho_v(h) & \psi(g) + g\psi(h) \\ \hline 0 & 1 \end{array}\right).$$

Hence we have an exact sequence $0 \to V \to E \to \mathbb{Q}_p \to 0$. What if we choose a different cocycle θ? We get an extension $0 \to V \to E' \to \mathbb{Q}_p \to 0$. As ψ and θ represent ϕ, we know there exists $a \in V$ such that $\forall g \in G_K$, we have $\theta(g) - \psi(g) = ga - a$. Then we define a map $E \to E'$ by $V \oplus \lambda \mapsto (V + \lambda a) \oplus \lambda$. This is G_K-equivariant isomorphism of extensions. □