Last time: $\mathbb{Z}G\text{-Mod} \to Ab$, $M \mapsto M^G$

l-exact

Is it right exact? No

$G = \mathbb{Z}$

$$0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$$

$1 \in G$ acts as: 1, $\begin{pmatrix}1&1\\0&1\end{pmatrix}$, 1

Invts $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0$ not exact.

<u>Lemma</u> C ab. cat.

A seq $0 \to Y \to Y' \to Y''$ is exact in C <u>if + only if</u> for every $X \in Obj(C)$,

$0 \to Hom_C(X, Y) \to Hom_C(X, Y') \to Hom_C(X, Y'')$ exact in <u>Ab</u>.

<u>Pf</u> "Only if" we know. For "if" direction use univ property of kernels.

<u>Corollary</u> C, D ab cats, $L : C \to D$, $R : D \to C$ additive functors st

$$Hom_D(LX, Y) \cong Hom_C(X, RY).$$

Then L is right-exact + R is left exact.

<u>Pf</u> By symmetry STP that R is left-exact.

Let $0 \to Y \to Y' \to Y''$ exact in D.

Let $X \in Obj\ C$.

$$\begin{array}{ccccc} 0 \to & Hom_C(X, RY) & \to & Hom(X, RY') & \to \cdots \\ & \wr| & & \wr| & \\ 0 \to & Hom_D(LX, Y) & \to & Hom_D(LX, Y') & \to \cdots \end{array}$$

Bottom row is exact ($Hom(LX, -)$ is a left-exact functor)

Hence so is top row. $\square$

Example: $\mathbb{Z}G\text{-Mod} \to Ab$, $M \mapsto M^G$ has a left adj^t. Thus it is left-exact.

<u>Projective + injective objects</u>

<u>Def</u> C ab. cat.

$X \in Ob(C)$ is <u>projective</u> if $Hom(X, -) : C \to Ab$ is exact.

X is <u>injective</u> if $Hom(-, X) : C^{op} \to Ab$ is exact. (equ^{tly}: if X is proj as an obj of C^{op}).

<u>Lemma</u>

(i) X is projective if "homs from X to quotients always lift,"

i.e. if $Y \to Y'$ has zero cokernel

$$\begin{array}{ccc} Y & \to & Y' \\ & \nwarrow & \nearrow \\ & X & \end{array}$$

any hom $X \to Y'$ lifts to $X \to Y$.

(ii) X is injective if "homs from subobjects to X always extend": if $0 \to Y \to Y'$, any hom $Y \to X$ extends to Y'.

$$\begin{array}{ccc} 0 \to Y & \to & Y' \\ \downarrow & & \\ X & & \end{array}$$

<u>Pf</u> We already know $Hom(X, -)$ left-exact, so it's exact if it sends surjections to surjections (+ dually).

<u>Examples</u>

- In R-Mod, R any ring, free modules are projective.

  If $X = \bigoplus_{i \in I} Re_i$, $Y \twoheadrightarrow Y'$, $f : X \to Y'$ is determined by $f(e_i)$ + can lift these to Y arbitrarily.

- In <u>Ab</u>, proj = free.

  In gen^l rings $\exists$ more projs, eg $R = k \oplus k$, k field. $M = k \oplus 0$ is proj but not free.

- <u>Thm</u> (Quillen–Suslin): Every proj. module over $k[X_1, \dots, X_n]$ is free. (conjectured by Serre)

- In <u>Ab</u>, the injective modules are divisible ab gps.

  To see why this is needed, consider

$$\begin{array}{ccc} 0 \to \mathbb{Z} & \xrightarrow{[2]} & \mathbb{Z} \\ {\scriptstyle id}\downarrow & \swarrow\!\!\!\!\times & \\ \mathbb{Z} & & \end{array}$$

doesn't extend!

Eg $\mathbb{Q}/\mathbb{Z}$ is divisible.

<u>Def^n</u> C has enough projectives if every obj is a quotient of a projective ($\forall X$ $\exists$ exact seq $P \to X \to 0$)

Similarly <u>enough injs</u> if $\forall X$ $0 \to X \to I$.

Eg in R-Mod, for any X, the free module on the underlying set of X is proj + surjects to X.

R-Mod <u>also</u> has enough injs (harder!)

<u>The category of complexes</u>

C ab. cat.

<u>Def^n</u> The category $Ch^\cdot(C)$ of <u>cochain complexes</u> over C is def. as follows:

- ob.is are pairs $((X^i)_{i \in \mathbb{Z}}, (d^i)_{i \in \mathbb{Z}})$ where $X^i \in Obj(C)$, $d^i \in Hom_C(X^i, X^{i+1})$

- morphisms are collections $(f^i)_{i \in \mathbb{Z}}$, $f^i \in Hom_C(X^i, Y^i)$ st $d^i_Y \circ f^i = f^{i+1} \circ d^i_X$ $\forall i$.

Frequently we denote objs by $X^\cdot$ (suppressing d^i from notation)

Also have $Ch_\cdot$ where differentials lower degree by 1 rather than raising.

(I'll emphasize use of $Ch^\cdot$ but be aware Weibel uses $Ch_\cdot$)

<u>CORRECTION</u> I forgot to say: objects of $Ch^\cdot(C)$ must satisfy $d^{i+1} \circ d^i = 0$ ie they <u>must be complexes</u>)

<u>Prop</u> $Ch^\cdot$ is abelian, & kernels + cokernels are what you expect ($\ker f^\cdot : X^\cdot \to Y^\cdot$ has i^{th} term $\ker(f^i)$ etc).

<u>Pf</u> Tedious check.

(<u>Warning</u> Not every complex in $Ch^\cdot(C)$ with inj/proj terms is inj/proj in $C^\cdot$ in gen^l.)

<u>Def^n</u> The "n^{th} cohomology functor" $Ch^\cdot(C) \to C$ sends $X^\cdot$ to n^{th} cohomology $\dfrac{\ker(d^n_X)}{\operatorname{im}(d^{n-1}_X)}$ (w. evident action on morphisms)

(Notation: $Y/X := \operatorname{coker}(X \to Y)$)

<u>Delicate point</u>

This is not actually well-def, because kernels + cokernels are not unique "on the nose".

Need to have a rule attaching a specific choice of kernel + cokernel to every morphism in C.

This is OK for R-Mod, but dangerous in gen^l, since morphisms in C won't be a set!

If have such a rule, then cohomology functors H^n are def.

<u>Def^n</u> A morphism in $Ch^\cdot(C)$ is a <u>quasi-isomorphism</u> if it induces isomorphisms on H^n $\forall n \in \mathbb{Z}$.

Note quasi-isos may not have inverses, eg

$$\begin{array}{ccccccc} X^\cdot & = & \cdots 0 \to & \mathbb{Z} & \xrightarrow{[2]} & \mathbb{Z} & \to 0 \cdots \\ \downarrow & & {\scriptstyle (0)} & \downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (2)} & \\ Y^\cdot & = & \cdots \to 0 \to & \mathbb{Z}/2 & \xrightarrow{[2]} & \mathbb{Z}/4 & \to 0 \cdots \end{array}$$

is a quasi-iso, but has no inverse.

NB: Not every collection of isos $H^i(X^\cdot) \xrightarrow{\sim} H^i(Y^\cdot)$ will come from a quasi-iso.

<u>Lemma</u> ("Snake Lemma")

Consider a diagram

$$\begin{array}{ccccccc} & A & \to & B & \to & C & \to 0 \\ & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow & & {\scriptstyle \gamma}\downarrow & \\ 0 \to & A' & \to & B' & \to & C' & \end{array}$$

with exact rows.

Then $\exists$ morphism $\operatorname{Ker}(\gamma) \to \operatorname{Coker}(\alpha)$ fitting into an exact seq

$$\ker\alpha \to \ker\beta \to \ker\gamma \xrightarrow{\delta} \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma.$$

and δ has a certain naturality property. (which I will not state precisely)

<u>Proof for R-Mod</u> (sketch)

Let $c \in C$ st $\gamma(c) = 0$. Then c is the image of some $b \in B$.

$\beta(b) \in B'$ maps to 0 in C' so $\beta(b) \in \operatorname{im}(A')$. Changing choice of b changes this elt of A' by an elt in image(A), so image in coker α is well-def.

(Weibel's book references a Hollywood film for this.)

For gen^l ab cats C:

- $\exists$ nice clean purely categorical pf by Bergman
- "Freyd-Mitchell embedding" – any small ab cat embeds in R-Mod some R
- consider homs from a gen'l X to the objs. $\square$