Today: Some examples of derived functors

Black: examples
Blue: gen^l properties of derived functors

Example 1 $\mathcal{C}$ any ab cat w. enough injs

$A \in \mathrm{Obj}\,\mathcal{C}$ $\quad \mathrm{Hom}(A,-) : \mathcal{C} \to \underline{Ab}$ left exact.

$$\mathrm{Ext}^i_{\mathcal{C}}(A,B) = R^i\big(\mathrm{Hom}_{\mathcal{C}}(A,-)\big)(B)$$

This is a functor $\mathcal{C}^{op} \times \mathcal{C} \to Ab$

If $A_1 \xrightarrow{f} A_2$ get a nat^l transf^n

$\mathrm{Hom}(A_2,-) \to \mathrm{Hom}(A_1,-)$

$\rightsquigarrow$ nat^l transf^{ns} between right derived functors

$\rightsquigarrow \mathrm{Ext}^i(A_2,B) \to \mathrm{Ext}^i(A_1,B)$.

We're using here the fact that a nat^l transf^n $F_1 \to F_2$ between left-exact functors yields nat^l transf^n's $R^i(F_1) \to R^i(F_2)$ $\forall i$.

Prop: TFAE for $A \in \mathrm{Ob}(\mathcal{C})$

- (1) A is projective
- (2) $\mathrm{Ext}^i(A,-)$ is zero functor $\forall i > 0$
- (3) $\mathrm{Ext}^1(A,-)$ is zero functor.

("Ext detects projectives")

Gen^l statement: TFAE

- F is exact
- $R^iF = 0$ $\forall i \geq 1$
- $R^1F = 0$

$1 \Rightarrow 2 \Rightarrow 3$ clear. If (3) holds, for $0 \to A \to B \to C \to 0$

$0 \to FA \to FB \to FC \to \cancel{R^1FA}$ so F exact.
$\quad =0$

$3 \Rightarrow 1$. $\square$

Example $\mathcal{C} = Ab$ $\quad n \geq 2$

$\mathrm{Ext}^i(\mathbb{Z}/n, \mathbb{Z})$? $\quad \mathrm{Ext}^0 = \mathrm{Hom}(-) = 0$

$\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is inj^{s} res^n of $\mathbb{Z}$

$\mathrm{Hom}(\mathbb{Z}/n, \mathbb{Q}) = 0$

$\mathrm{Hom}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$? $1 \in \mathbb{Z}/n$ must go to a/n some a
$\cong \mathbb{Z}/n$.

so $\mathrm{Ext}^1 = \mathbb{Z}/n$
$\mathrm{Ext}^j = 0$ $j \neq 1$

So Ext^i are coho. of complex
$0 \cdots 0 \to \mathbb{Z}/n \to 0 \to \cdots$

We can consider $\mathrm{Hom}(-,B) : \mathcal{C}^{op} \to \underline{Ab}$

If $\mathcal{C}^{op}$ has enough injs, can derive this too.

Prop If $\mathcal{C}$ has both enough injs & enough projs, then

$$\mathrm{Ext}^i(A,B) = R^i\big(\mathrm{Hom}(-,B)\big)(A)$$

Equiv^{ly}: can compute Ext using <u>either</u> an inj res^n of B <u>or</u> a proj res^n of A.

Eg: can use $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$ (proj res^n of $\mathbb{Z}/n$) in above example.

("Balancing" of Ext functor.)

Sketch pf Choose $P_\cdot \to A$ proj res^n
$B \to I^\cdot$ inj res^n

Want to show $H^i(\mathrm{Hom}(P_\cdot, B))$
$\cong H^i(\mathrm{Hom}(A, I^\cdot))$

Consider the gps $X^{pq} = \mathrm{Hom}(P_p, I^q)$

$$T^i = \bigoplus_{p+q=i} X^{pq} \quad \text{(total complex)}$$

Fact: $T^\bullet$ is a cochain complex, and the maps $P_0 \to A$, $B \to I^0$ give quasi-isos

$\mathrm{Hom}(A, I^\cdot) \xrightarrow{\sim} T^\cdot \xleftarrow{\sim} \mathrm{Hom}(P_\cdot, B)$. $\square$

Corollary B is injective

$\Leftrightarrow \mathrm{Ext}^i(A,B)$ vanishes $\forall A$ $\forall i \geq 1$

$\Leftrightarrow \mathrm{Ext}^1(A,B)$ "

Example 2 Group cohomology

G group $\rightsquigarrow \mathbb{Z}G\text{-Mod}$

$(-)^G : \mathbb{Z}G\text{-Mod} \to Ab$.

nat^{ly} isomorphic to $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z},-)$.

Def^n $H^i(G,M) := R^i\big((-)^G\big)(M)$
$= \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, M)$.

2 approaches to computing these:

- inj res^n of M
- proj res^n of $\mathbb{Z}$. $\leftarrow$ less work!

Eg: $G = \infty$ cyclic gp generator g.

Then $\mathbb{Z}G \cong \mathbb{Z}[X, X^{-1}]$

$\mathbb{Z}G \xrightarrow{g-1} \mathbb{Z}G$

is a proj res^n of $\mathbb{Z}$ in $\mathbb{Z}G$-Mod

So for any $M \in \mathbb{Z}G\text{-Mod}$,

$H^i(G,M) = i^{th}$ coho of

$\mathrm{Hom}(\text{this}, M)$

$= M \xrightarrow{[g-1]} M \qquad H^0(G,M) = M^{g=1}$

$H^1(G,M) = M/(g-1)M$.

For any G, $\exists$ systematic way to build a proj res^n of $\mathbb{Z}$ in $\mathbb{Z}G$-Mod ("bar resolution").

Def X_n = free $\mathbb{Z}G$-module on set of symbols $(g_1, \dots, g_n)$ $g_i \in G$

$X_0 = \mathbb{Z}G$

$d : X_n \to X_{n-1}$ def. as $\sum_{i=0}^{n} (-1)^i d^{(i)}$

$d^{(0)}((g_1 \cdots g_n)) = [g_1] \cdot (g_2 \cdots g_n)$

$d^{(i)}((g_1 \cdots g_n)) = (g_1, \dots, g_i g_{i+1}, g_{i+2} \cdots g_n)$

$d^{(n)}(\sim\sim) = (g_1 \cdots g_{n-1})$.

Eg $X_0 = \mathbb{Z}G$ $\qquad d : X_1 \to X_0$ sends

$X_1 = \bigoplus_{g \in G} \mathbb{Z}G \cdot (g)$ $g \in G$ $\qquad (g) \mapsto g-1$.

Fact $X_\cdot$ is a proj (indeed free) res^n of $\mathbb{Z}$.

Hence $H^i(G,M) = H^i(\mathrm{Hom}(X_\cdot, M))$,

$\mathrm{Hom}(X_i, M)$ has a name: it's the gp of <u>i-cochains</u> on G w. values in M

$=$ M-valued fcns on $\underbrace{G \times \cdots \times G}_{i \text{ times}}$.

$C^i(G,M) := \mathrm{Hom}_{\mathbb{Z}G}(X_i, M)$

$$H^i(G,M) = H^i(C^\cdot(G,M))$$

In many cases can compute derived functors using a much wider class of resolutions.

$F : \mathcal{C} \to \mathcal{D}$ left-exact, $\mathcal{C}$ enough injs

Def^n $Y \in \mathrm{Obj}(\mathcal{C})$ is <u>F-acyclic</u> if $R^i(F)(Y) = 0$ $\forall i \geq 1$.

(Inj objs are F-acyclic for <u>every</u> F.)

Prop Let $X \in \mathrm{Obj}\,\mathcal{C}$ and $[X] \to Y^\cdot$ an F-acyclic right resolution of X.

Then $R^i(F)(X) \cong H^i(F(Y^\cdot))$.

Proof We have, for all $n \geq 0$, SESs

$0 \to Z^n(Y^\cdot) \to Y^n \to Z^{n+1}(Y^\cdot) \to 0$.

(because $Y^\cdot$ exact at $n+1$ spot.)

LES:

$0 \to F(Z^nY) \to F(Y^n) \to F(Z^{n+1}Y)$
$\to R^1F(Z^nY) \to \cancel{R^1F(Y^n)} \to R^1F(Z^{n+1}Y)$
$\qquad\qquad\qquad\qquad 0$
$\to \cdots \to 0 \to \cdots$

$\Rightarrow$ for all $n \geq 0$, $j \geq 1$,

$R^j(F)(Z^{n+1}Y) \cong R^{j+1}(F)(Z^nY)$.

Hence $R^m(F)(X)$

$\cong R^mF(Z^0Y)$

$\cong R^{m-1}(F)(Z^1Y)$

$\vdots$

$\cong R^1(F)(Z^{m-1}Y)$

$= \mathrm{coker}\big(F(Y^{m-1}) \to F(Z^mY)\big)$

$= \mathrm{coker}\big(F(Y^{m-1}) \to Z^m(FY^\cdot)\big)$ (left exactness)

$= H^m(FY^\cdot)$ $\square$

(Notational point: if $(Y^\bullet, d^\cdot)$ is a cochain complex,

$Z^iY^\cdot = \ker(d^i : Y^i \to Y^{i+1})$ i-cocycles

$B^iY^\cdot = \mathrm{im}(d^{i-1} : Y^{i-1} \to Y^i)$ i-coboundaries

In topology: for $\mathcal{X}$ a simplicial complex,

$C^i(\mathcal{X}, \mathbb{Z}) = \{\mathbb{Z}\text{-valued fcns on } i\text{-simplices of } \mathcal{X}\}$

+ cocycles + coboundaries of this cplx are the terms as def. in topology)