DERIVED CATEGORIES

Refs: · Weibel ch. 9
· Gelfand–Manin
· Holm, Jørgensen, Rouquier "Triangulated Categories"

$\mathcal{C}$ ab. cat. $Ch(\mathcal{C})$ cochain complexes

Def $K(\mathcal{C})$ – cat w. same objs as $Ch(\mathcal{C})$, morphisms = <u>homotopy classes</u> of cochain maps.

This is a well-def. additive cat (tiny check) with a canonical additive functor $Ch^{\cdot}(\mathcal{C}) \to K(\mathcal{C})$.

Similarly $K_+(\mathcal{C})$ (bounded below complexes)
$K_-(\mathcal{C})$ (bounded above)
$K_b(\mathcal{C})$ (bounded)

$K_? =$ one of K, K_+, K_-, K_b.

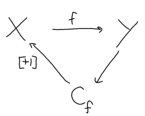
· Cohomology functors $H^i : Ch_?(\mathcal{C}) \to \mathcal{C}$ factor thru $K_?(\mathcal{C})$.

· For any additive $F: \mathcal{C} \to \mathcal{D}$ we get a functor $F: K_?(\mathcal{C}) \to K_?(\mathcal{D})$.

Fact If $X \in Ob(\mathcal{C})$ has any injective res^n, then it's unique up to unique isomorphism as an obj of $K_+(\mathcal{C})$. Sim. proj res's unique in $K_-(\mathcal{C})$.

Problem The $K_?(\mathcal{C})$ are <u>not</u> abelian (except in trivial cases). (cf H–J–R)
– some morphisms have <u>no kernels</u>.
Substitute for kernels + cokernels: <u>mapping cones</u>.

$f: X \to Y$ cochain map
C_f cone of f.



Composite of any 2 morphisms is 0 in $K_?(\mathcal{C})$

Long exact seq $H^i(X) \xrightarrow{f} H^i(Y) \to H^i(C_f) \to H^{i+1}(X) \to \cdots$

"Def" A <u>triangulated cat</u> is an additive cat with shift operators $[m]$, $m \in \mathbb{Z}$, and a class of <u>distinguished triangles</u>
$X \to Y \to Z \to X[1]$ satisfying some axioms.

Axioms are chosen st the $K_?(\mathcal{C})$ are triangulated, with the obvious shift operators + distinguished triangles being anything isomorphic to a mapping cone triangle.

<u>Triangulated functor</u> $\mathcal{T}_1 \to \mathcal{T}_2$
= additive functor commuting w. shifts + sending dist. tri.s to dist. tris.

<u>Cohomological functor</u> $F: \mathcal{T} \to \mathcal{A}$ (tri) (ab)
is an additive functor st $\forall$ dist. tris $X \to Y \to Z$
we get a long exact seq

$$\cdots \to F(X) \to F(Y) \to F(Z) \to F(X[1]) \to \cdots$$

Eg: · cohomology $F(X) = H^0(X)$
$F(X[n]) = H^n(X)$
for $\mathcal{T} = K_?(\mathcal{C})$.

· for any $\mathcal{T}$, any $A \in Ob(\mathcal{T})$
$Hom_{\mathcal{T}}(A, -)$ is a coho. functor $\mathcal{T} \to \underline{Ab}$. (nonobvious consequence of the axioms I didn't tell you.)

(Exercise: If $\mathcal{T} = K_?(\underline{Ab})$ then
$H^0(X) = Hom_{\mathcal{T}}([\mathbb{Z}], X)$.
So cohomology is a <u>representable</u> functor on $K_?(\underline{Ab})$.)

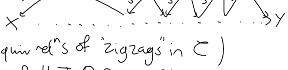
<u>Localisation of categories</u>

Going from $Ch(\mathcal{C})$ to $K(\mathcal{C})$ we made more morphisms into isomorphisms.

$\exists$ a notion of <u>localisation of cats</u>:
$\mathcal{C}$ cat, S class of morphisms in $\mathcal{C}$, the localisation of $\mathcal{C}$ at S is a category D with a functor $F: \mathcal{C} \to D$ st
- $F(s)$ is an isomorphism $\forall s \in S$
- if $\mathcal{C} \xrightarrow{G} E$ also makes all $s \in S$ into isomorphisms, then G factors thru F.

<u>Thm</u> If $Ob(\mathcal{C})$ is a set, then all localisations of $\mathcal{C}$ exist.

Idea of construction: D has same obs as $\mathcal{C}$, + morphisms look like diagrams



(equiv rel^{n}s of "zigzags" in $\mathcal{C}$)
— cf. H-J-R for a careful description.

Want: $D_?(\mathcal{C})$ $\mathcal{C}$ abelian

should be localisation of $Ch_?(\mathcal{C})$ at quasi-isomorphisms.

We'll use $K_?(\mathcal{C})$ as a stepping stone.
Then we get for free that $Ch_?(\mathcal{C}) \to D_?(\mathcal{C})$ factors thru $K_?(\mathcal{C})$ (not obvious a priori.)

Useful Fact: Any "zigzag" in $K(\mathcal{C})$, S = q-isos, can be shrunk down to length 1.

So morphisms in $D_?(\mathcal{C})$ are "roofs"  in $K(\mathcal{C})$.

Consequence: $D_?(\mathcal{C})$, if it exists (!) is triangulated, + $K_? \to D_?$ is a tri. functor.

Prop If $\mathcal{C}$ has enough injs, $D_+(\mathcal{C})$ exists + it's equivalent to subcat of $K_+(\mathcal{C})$ consisting of complexes of injective objects of $\mathcal{C}$.
Dually if enough projs $D_-(\mathcal{C})$ exists + is equiv to K_-(proj obs of $\mathcal{C}$).

Existence of CE res's shows that any obj of $Ch_+(\mathcal{C})$ (or $K_+(\mathcal{C})$) is quasi-iso to a complex of injectives.
Tedious check: morphisms in $D_+(\mathcal{C})$ between inj. complexes are the same as in $K_+(\mathcal{C})$.

Upshot If $F: \mathcal{C} \to \mathcal{D}$ additive, can define $RF: D_+(\mathcal{C}) \to D_+(\mathcal{D})$ to be the functor agreeing with F on complexes of injectives.

If F is left-exact, then $H^i(RF): D_+(\mathcal{C}) \to \mathcal{D}$ is the i^{th} hyperderived functor $\mathbb{R}^i(F)$. But RF exists w/o assuming left-exactness.
We want to focus on RF, + regard the R^iF as "approximations".

Eg if G group, M a G-module,

$R(G,M)$ = obj in $D_+(Ab)$

whose cohomology gps are $H^i(G,M)$, but it contains more info, eg if $H \trianglelefteq G$ can recover $R(G,M)$ on the nail from $R(H,M)$:

$$\boxed{R(G/H, R(H,M)) = R(G,M)}$$

(Similar picture for total left derived functors when enough projectives.)

Some constructions in $D_?(R\text{-Mod})$

- for R commutative, tensor product $R\text{-Mod} \times R\text{-Mod} \to R\text{-Mod}$ extends to derived tensor product

$$D_-(R\text{-Mod}) \times D_-(R\text{-Mod}) \to D_-(R\text{-Mod})$$
$$A \ , \ B \longrightarrow A \otimes^{\mathbb{L}} B$$

If A, B are concentrated in deg 0, then $H^{-i}(A \otimes^{\mathbb{L}} B) = \mathrm{Tor}_i^R(A,B)$.

- derived Hom (RHom)

$$D_-(R\text{-Mod})^{op} \times D_+(R\text{-Mod}) \to D_+(R\text{-Mod})$$

$$A, B \mapsto RHom(A, B)$$

If A, B R-modules in degree 0, then

$$H^i(RHom(A,B)) = Ext^i_R(A,B)$$

- X top. space, $\exists$ object $C_\cdot(X) \in D_-(Ab)$ whose cohomology is $H_{-i}(X, \mathbb{Z})$.

Similarly $C^\cdot(X)$ cohomology complex

$$C^\cdot(X) = RHom(C_\cdot(X), \mathbb{Z})$$

$C_\cdot(X, A)$ A ab. gp.

$$= C_\cdot(X) \otimes^{L} A.$$

- Serre / Grothendieck duality

Serre duality: X/k smooth proj. alg variety dim^n d

$\mathcal{F}$ coherent sheaf on X (eg. sections of a vector bundle $\mathcal{L}$)

$H^i(X, \mathcal{F})$ finite-dim^l / k.

$$H^{d-i}(X, \mathcal{F})^\vee = H^i(X, \underbrace{D(\mathcal{F})}_{\text{'dual sheaf'}})$$

($d = \dim X$). If $\mathcal{F}$ = sections of $\mathcal{L}$

$$D(\mathcal{F}) = (\text{sections of } \mathcal{L}^\vee) \otimes \underbrace{\omega_X}_{\text{line bundle}}.$$

If X not smooth this doesn't work.

Grothendieck, Verdier: $\exists$ duality functor on D_b(coh. sheaves on X) — only exists in derived cat.

($\exists$ analogues for sheaves of ab gps on locally compact spaces, or étale sheaves. — "six operations")