There are natural transformations $F\circ G \longrightarrow id_{AbGrp}$ and $id_{Set} \longrightarrow G\circ F$
$G\circ F \to$

Adjoint functors

In the example above we have a bit more structure. If S is any set and T any abelian group then

$$(\text{maps of set from } S \text{ to } T) = (\text{maps of abelian groups } F(S) \to T)$$

because any map of sets $S \to T$ can be uniquely extended to a group morphism $F(S) \to T$. These are natural:

$$Hom_{Set}(S, G(T)) \xrightarrow{\simeq} Hom_{AbGrp}(F(S), T)$$

agree as functors

$$\begin{aligned} Set^{op} \times AbGrp &\longrightarrow Set \\ (S,T) &\longmapsto Hom_{Set}(S, GT) \\ &\longmapsto Hom_{Grp}(FS, T) \end{aligned}$$

this structure is called an adjunction

Let $\mathcal{C},\mathcal{D}$ categories and $\begin{matrix} F: \mathcal{C}\to\mathcal{D} \\ G: \mathcal{D}\to\mathcal{C}\end{matrix}$ functors. they are adjoint if the functors

$$\begin{aligned} \mathcal{C}^{op}\times\mathcal{D} &\longrightarrow Set \\ (X,Y) &\longmapsto Hom_{\mathcal{D}}(FX,Y) \\ &\longmapsto Hom_{\mathcal{C}}(X,GY) \end{aligned}$$

are naturally isomorphic

We say that F is left adjoint to G and G is right adjoint to F

Exercise: 1) Show that the forgetful functor $Top \to Set$ has a left adjoint and describe it explicitly
2) Does it have a right adjoint?

Def: an additive category is a category $\mathcal{C}$ together with a binary operation $+$ on every Hom-set $Hom_{\mathcal{C}}(X,Y)$ making it into an abelian group for every $X,Y \in Ob(\mathcal{C})$ such that the following axioms are satisfied

- A2 · $\exists\, 0 \in Ob(\mathcal{C})$ zero object such that $Hom_{\mathcal{C}}(X,0)$ = trivial abelian group $\forall$, $Hom_{\mathcal{C}}(0,X)$ - trivial abelian group $\forall X$
- A1 · composition of morphisms is bilinear
- A3 · any two objects $X,Y \in Ob(\mathcal{C})$ have a direct sum $X\oplus Y$ satisfying the universal property.

The universal property of the direct sum is:

$$\begin{tikzcd} X_1 \arrow[d, "id"'] \arrow[dr, "i_1"] & & X_2 \arrow[dl, "i_2"'] \arrow[d, "id"] \\ X_1 & X_1\oplus X_2 \arrow[l, "p_1"] \arrow[r, "p_2"'] & X_2 \end{tikzcd}$$

the diagram is commutative and the diagonal compositions are zero.

and

- for any $Z \in Ob(\mathcal{C})$ and for any homomorphisms $\begin{matrix} Z \to X_1 \\ Z \to X_2 \end{matrix}$ there exists a unique homomorphism $Z \to X_1\oplus X_2$ such that

$$\begin{tikzcd} & & X_1 \\ Z \arrow[r, dashed, "\exists!"] \arrow[urr, bend left] \arrow[drr, bend right] & X_1\oplus X_2 \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\ & & X_2 \end{tikzcd}$$

commutes

- for any $Z \in Ob(\mathcal{C})$ and morphisms $X_1 \to Z$, $X_2 \to Z$ there exists a unique morphism $X_1\oplus X_2 \to Z$ s.t. the analogue diagram commutes.

In particular, the direct sum is both a product (because it satisfies the second property) and a coproduct (because it satisfies the first one) in the category $\mathcal{C}$. The universal properties specify $X_1 \oplus X_2$ up to unique isomorphisms.

Def: an additive functor ~~fro~~ between two additive categories $\mathcal{C}, \mathcal{D}$ is a functor $\mathcal{C} \to \mathcal{D}$ such that the map

$$F : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(FX, FY)$$

is a group homomorphism for every $X, Y \in \mathrm{Ob}(\mathcal{C})$.

An additive functor automatically sends $0_{\mathcal{C}}$ to $0_{\mathcal{D}}$ and respects direct sums.

EXAMPLES

- AbGrp · The category of groups is not additive; notice that in Grp products and coproducts are different
- $\mathrm{Vect}_{\mathbb{R}}$ (or Vect_k for any field k)
- $\mathrm{Ban}_{\mathbb{C}}$ complex Banach spaces with continuous (bounded) linear maps

  There are some choices in defining the norm on $X \oplus Y$, e.g. $\|\cdot\|_X + \|\cdot\|_Y$ or $\max\{\|\cdot\|_X, \|\cdot\|_Y\}$. They give rise to equivalent norms (i.e. isomorphic objects) that are not the same (i.e., the objects are different).

If $\mathcal{C}$ is an additive category we have a notion of zero morphism and we use it in the following definitions.

Def: let $\mathcal{C}$ be an additive category and $\varphi : X \to Y$ a morphism in $\mathcal{C}$. A kernel of φ is an object K together with a morphism $K \xrightarrow{\kappa} X$ such that

1) $$\begin{tikzcd} K \arrow[r, "\kappa"] \arrow[rr, bend right, "0"'] & X \arrow[r, "\varphi"] & Y \end{tikzcd}$$ the composition $\varphi \circ \kappa = 0$

2) Given L, $l : L \to X$ such that $\varphi \circ l$ is 0, then there exists a unique morphism $L \to K$ such that

$$\begin{tikzcd} L \arrow[d, dashed, "\exists!"'] \arrow[dr, "l"] & & \\ K \arrow[r, "\kappa"] & X \arrow[r, "\varphi"] & Y \end{tikzcd}$$ commutes

Kernels are unique up to unique isomorphisms if they exist.

Def: cokernels are kernels in the opposite category. Formally, a cokernel of $\varphi : X \to Y$ is a pair $(C, c : Y \to C)$ s.t.
~~$c \circ \varphi = 0$~~

1) $$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[rr, bend right, "0"'] & Y \arrow[r, "c"] & C \end{tikzcd}$$ the composition $c \circ \varphi = 0$

2) Given D, $d : Y \to D$ such that $d \circ \varphi = 0$, then there exists a unique morphism $C \to D$ such that

$$\begin{tikzcd} & & D \\ X \arrow[r, "\varphi"] & Y \arrow[ur, "d"] \arrow[r, "c"] & C \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$ commutes

Note that $\mathcal{C}^{op}$ is additive if $\mathcal{C}$ is, kernels in $\mathcal{C}$ are cokernels in $\mathcal{C}^{op}$ and viceversa.

Def: let $\mathcal{C}$ be an additive category with kernels and cokernels. Let $\varphi : X \to Y$, then the Image of φ is

$$\mathrm{Im}(\varphi) = \ker(Y \longrightarrow \mathrm{Coker}\,\varphi)$$

and the coimage is

$$\mathrm{Coim}(\varphi) = \mathrm{Coker}(\ker\varphi \longrightarrow X)$$

In the above situation, there is a unique map $\mathrm{Coim}(\varphi) \to \mathrm{Im}(\varphi)$ such that

$$X \longrightarrow \mathrm{Coim}(\varphi) \longrightarrow \mathrm{Im}(\varphi) \longrightarrow Y \qquad \text{is } \varphi.$$

Def: we say that $\mathcal{C}$ is an abelian category if it is an additive category where cokernels and kernels exists and $\mathrm{Coim}(\varphi) \to \mathrm{Im}(\varphi)$ is an isomorphism for every φ.

The naive intuition for this definition is that we want to consider categories where the first isomorphism theorem holds, which is exactly the statement that $\mathrm{Coim}(\varphi) \cong \mathrm{Im}(\varphi)$ $\forall \varphi$.

EXAMPLES

- AbGrp is abelian (first isom. theorem)
- R-Mod for any ring R

Abelian categories are the right settings for homological algebra. Indeed, as we will see later, their structure gives the possibility to do (homological) algebra and allows several construction.
For instance, notice that if in an abelian category a morphism φ has $\ker\varphi = 0 = \mathrm{coker}\,\varphi$, then

$$\mathrm{Im}(\varphi) = \ker(Y \to 0) = Y \qquad \mathrm{Coim}(\varphi) = \mathrm{coker}(\underbrace{\ker\varphi}_{0} \to X) = X$$

then

$$\begin{array}{c}\mathrm{Coim}(\varphi) \longrightarrow \mathrm{Im}(\varphi) \\ \| \qquad\qquad \| \\ X \xrightarrow{\ \varphi\ } Y\end{array}$$

is an isomorphism. Thus we have a criterion to check whether φ is an isomorphism or not (which is more useful than to show the existence of an inverse).

Exact Sequences

Def: let $\mathcal{C}$ be an abelian category and

$$\cdots \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow \cdots$$

a sequence of objects and morphisms. We say this sequence is a complex if the composition of any two consecutive morphisms (arrows) is 0.
We say it is exact at Y is the natural map $\mathrm{Im}(X \to Y) \to \ker(Y \to Z)$, which exists because $X \to Y \to Z$ is 0, is an isomorphism.
We say the sequence is exact if it is exact at every object in the sequence (except end points).

A short exact sequence is a five-terms sequence with endpoints zero $0 \to X \to Y \to Z \to 0$.

Def: let $\mathcal{C}, \mathcal{D}$ be abelian categories, let $F: \mathcal{C} \to \mathcal{D}$ additive functor. We say F is exact if, for all short exact sequence $0 \to X \to Y \to Z \to 0$, the sequence

$$0 \longrightarrow FX \longrightarrow FY \longrightarrow FZ \longrightarrow 0$$

is still a short exact sequence.
We say F is

- left exact if $0 \to FX \to FY \to FZ$ is exact
- right exact if $FX \to FY \to FZ \to 0$ is exact

EXAMPLES

- the forgetful functor $\mathrm{Vect}_k \to \mathrm{AbGrp}$ is exact (and also R-Mod $\to$ AbGrp)
- the tensor product is a right exact functor, the Hom-functor is left exact (for example in R-Mod).

Fiddly Lemmas: 1) If F is left exact, then for any $0 \to X \to Y \to Z$ exact in $\mathcal{C}$, $0 \to FX \to FY \to FZ$ is exact in $\mathcal{D}$.

2) Similarly, if F is right exact, for any $X \to Y \to Z \to 0$ exact in $\mathcal{C}$, $FX \to FY \to FZ \to 0$ is exact in $\mathcal{D}$.

(this is exercise 1.6.3 in Weibel's book)

Proof (Sketch): It suffices to do (1), since (2) is (1) in the opposite categories. Assume

$$0 \to X \xrightarrow{\varphi} Y \to Z \quad \text{exact}$$

then

$$0 \to X \to Y \to \operatorname{Coker} \varphi \to 0 \quad \text{is exact}$$

$$0 \to \operatorname{Coker} \varphi \to Z \to W \to 0 \quad \text{is exact}, \quad W = \operatorname{coker}(Y \to Z)$$

left exactness gives

$$0 \to FX \to FY \to F(\operatorname{Coker} \varphi)$$
$$0 \to F(\operatorname{Coker} \varphi) \to FZ \to FW$$

the map $F(\operatorname{coker} \varphi) \to FZ$ has zero kernel, consequently $FY \to F(\operatorname{coker} \varphi)$, $FY \to FZ$ have the same kernel.
Thus $0 \to FX \to FY \to FZ$ is exact.

□

It follows that left exact functors preserve kernels: for any φ the sequence $0 \to \ker \varphi \to X \to Y$ is exact, then $F(\ker \varphi) = \ker F\varphi$. Similarly right exact functors preserve cokernels.
Actually, this last remark can be reversed, i.e. a functor that preserves kernels is left exact.

Prop: let $X \in Ob(\mathcal{C})$. Then the functor

$$\mathcal{C} \to AbGrp \qquad Y \longmapsto \operatorname{Hom}_{\mathcal{C}}(X,Y)$$

is left exact.

Proof: let $0 \to Y \xrightarrow{\alpha} Y' \xrightarrow{\beta} Y''$ be exact. We are claiming that

$$0 \to \operatorname{Hom}(X,Y) \xrightarrow{\alpha^*} \operatorname{Hom}(X,Y') \xrightarrow{\beta^*} \operatorname{Hom}(X,Y'')$$

is exact, i.e.
- $\operatorname{Hom}(X,Y) \to \operatorname{Hom}(X,Y')$ is injective
- if $g \in \operatorname{Hom}(X,Y')$ and $\beta \circ g = 0$, then $\exists f : X \to Y$ such that $g = \alpha \circ f$

Notice that $\beta^* \circ \alpha^* = 0$ automatically because functors preserve composition.
For the first one, if $f \in \operatorname{Hom}(X,Y)$ and $\alpha \circ f = 0$ then f factors through $\ker(\alpha)$, which is zero. Then $f = 0$.
For the second one, if $g \in \operatorname{Hom}(X,Y')$ and $\beta \circ g = 0$ then g factors uniquely through $\ker(\beta)$ by the universal property.
But $\ker(\beta) = Y$ by exactness. Hence there exist a unique $f \in \operatorname{Hom}(X,Y)$ such that $\alpha \circ f = g$.

□

Similarly, $\operatorname{Hom}(\cdot, X)$, as a functor $\mathcal{C}^{op} \to AbGrp$, is left exact. It can also be regarded as a contravariant functor from $\mathcal{C}$ to $AbGrp$, and it sends

sequences in $\mathcal{C}$ $\quad Y \to Y' \to Y'' \to 0 \quad$ (which is $0 \to Y'' \to Y' \to Y$ in $\mathcal{C}^{op}$)

to sequences $\underset{0}{\underset{\shortparallel}{\operatorname{Hom}(0,X)}} \to \operatorname{Hom}(Y'',X) \to \operatorname{Hom}(Y',X) \to \operatorname{Hom}(Y,X)$

EXAMPLES

$\mathcal{C} = \mathbb{Z}[G]$-modules $\quad$ = abelian groups with a linear G-action (G is a group)

$\mathbb{Z}[G]$ denotes the group ring built from G

Then there is an additive functor $\mathcal{C} \to AbGrp \qquad M \longmapsto M^G$ = G-invariants of M

Claim: this is left exact, but not exact.

For left exactness, $M \mapsto M^G$ is actually naturally isomorphic to $\mathrm{Hom}_{\mathbb{Z}[G]\text{-Mod}}(\mathbb{Z}, M)$. Exactness on the right does not hold (exercise: find a counterexample).