TCC Homological Algebra: Assignment #2

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This is the second of 4 problem sheets. Solutions should be submitted to me (via any appropriate method) by **noon on 16th November**. This problem sheet will be marked out of a total of 25; the number of marks available for each question is indicated.

Note that rings are assumed to be unital (i.e. having a multiplicative identity element 1), ring homomorphisms are assumed to map 1 to 1, and modules are left modules, unless otherwise stated.

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1. [3 points] For each of the following functors, determine whether it is exact, right exact, left exact, or neither. (Here G denotes an arbitrary group.)
   (a) The forgetful functor $\mathbf{Z}[G]$-$\underline{\text{Mod}} \to \underline{\text{Ab}}$.
   (b) The functor $\mathbf{Z}[G]$-$\underline{\text{Mod}} \to \underline{\text{Ab}}$ sending M to the G-coinvariants M_G (the quotient of M by the subgroup generated by $\{gm - m : g \in G, m \in M\}$.

2. Let $\text{Pro}(\underline{\text{Ab}})$ be the category whose objects are sequences $(C_i)_{i \geq 0}$ of abelian groups with homomorphisms $f_i : C_{i+1} \to C_i$ for every $i \geq 0$, with morphisms $(C_i) \to (D_i)$ being collections of morphisms $C_i \to D_i$ for every i commuting with the f_i.

   (You may assume that a sequence $0 \to (A_i) \to (B_i) \to (C_i) \to 0$ is exact in $\text{Pro}(\underline{\text{Ab}})$ if and only if $0 \to A_i \to B_i \to C_i \to 0$ is exact for every i.)
   (a) [1 point] Show that the functor $\text{Pro}(\underline{\text{Ab}}) \to \underline{\text{Ab}}$ mapping (C_i) to the inverse limit $\varprojlim_i C_i$ is left-exact.
   (b) [1 point] Give an example to show that it is not exact. *(Hint: Contemplate the sequence $\dots \longrightarrow \mathbf{Z} \xrightarrow{[2]} \mathbf{Z} \xrightarrow{[2]} \mathbf{Z}$.)*

3. [2 points] Show that $\mathbf{Z}/3\mathbf{Z}$ is projective as a module over $\mathbf{Z}/6\mathbf{Z}$.

4. [3 points] Let $R = \mathbf{Q}[X, Y, Z]$, and I the ideal (X, Y, Z) of R. Show that the complex

$$\dots 0 \longrightarrow R \xrightarrow{\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} 0 & -Z & Y \\ Z & 0 & -X \\ -Y & X & 0 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} X & Y & Z \end{pmatrix}} R \longrightarrow 0$$

   is a projective resolution of R/I.

5. [4 points] Let R be a ring.
   (a) Let $R^\vee = \text{Hom}_{\underline{Ab}}(R, \mathbf{Q}/\mathbf{Z})$, considered as an R-module via $(r \cdot \phi)(s) = \phi(sr)$. Show that $\text{Hom}_R(M, R^\vee) = \text{Hom}_{\text{Ab}}(M, \mathbf{Q}/\mathbf{Z})$ for any R-module M (naturally in M). Hence show that $R^\vee$ is an injective R-module.
   (b) Show that for any R-module M and any non-zero $m \in M$, there is a homomorphism $\psi : M \to R^\vee$ such that $\psi(m) \neq 0$.
   (c) Show that if X is a set and $(I_x)_{x \in X}$ are injective R-modules, then $\prod_{x \in X} I_x$ is injective.
   (d) Hence deduce that $\underline{R\text{-Mod}}$ has enough injectives.

6. [2 points] Give a detailed proof of the existence of the chain homotopy $(s^i)_{i\in\mathbf{Z}}$ in the uniqueness proposition for injective resolutions.

7. [2 points] *The Hom complex.* Let $X^\bullet, Y^\bullet$ be two cochain complexes over some abelian category $\mathcal{C}$. Let $\underline{Hom}(X^\bullet, Y^\bullet)$ be the complex of abelian groups with i-th term $\prod_{j\in\mathbf{Z}} \mathrm{Hom}_{\mathcal{C}}(X^j, Y^{i+j})$, and differential given by
$$d^i\left((f^j)_{j\in\mathbf{Z}}\right) = \left(d_Y^{i+j}\circ f^j + (-1)^{i+1} f^{j+1}\circ d_X^j\right)_{j\in\mathbf{Z}}.$$
Check that this is a cochain complex. Show that the 0-cocycles in this complex are the cochain maps $X^\bullet \to Y^\bullet$, and the 0-coboundaries are the null-homotopic ones.

8. [4 points] *Mapping cones.* Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism of cochain complexes (over some abelian category $\mathcal{C}$). The **mapping cone** of f is the complex $C_f^\bullet$ defined as follows:

   - C_f^i is the direct sum $X^{i+1}\oplus Y^i$;
   - the differential $d^i : C_f^i \to C_f^{i+1}$ is given by[1] $\begin{pmatrix} -d_X^{i+1} & 0 \\ f^{i+1} & d_Y^i \end{pmatrix}$.

   (a) Show that $C_f^\bullet$ is a cochain complex.

   (b) Let $X[1]^\bullet$ denote the cochain complex with i-th term X^{i+1}, and with differential $d_{X[1]}^i = -d_X^{i+1}$. Show that there is a short exact sequence of cochain complexes $0 \to Y^\bullet \to C_f^\bullet \to X^\bullet[1] \to 0$.

   (c) Assume that $\mathcal{C} = \underline{R\text{-Mod}}$ for some ring R. Show that in the long exact sequence of cohomology attached to the SES of complexes in (b), the coboundary map $\partial^i : H^i(X^\bullet[1]) = H^{i+1}(X) \to H^{i+1}(Y)$ coincides with the map $H^{i+1}(f^\bullet)$.

9. [3 points] Let $C^\bullet$ be the mapping cone of $f^\bullet : X^\bullet \to Y^\bullet$, as before.

   (a) Show that a morphism of complexes $g : W^\bullet \to X^\bullet$ factors through the canonical map $C_f^\bullet[-1] \to X$ if and only if $f\circ g$ is null-homotopic.

   (b) Show that a morphism of complexes $g : Y^\bullet \to Z^\bullet$ factors through the canonical map $Y^\bullet \to C_f^\bullet$ if and only if $g \circ f$ is null-homotopic. *(Hint: Look for a way to deduce this from part (a).)*

[1] Here we're identifying a morphism $F : A_1\oplus A_2 \to B_1 \oplus B_2$ with a 2×2 matrix whose i, j term is the morphism $F_{pq} : A_q \to B_p$ given by $\mathrm{pr}_p \circ F \circ \iota_q$. Note that some books use slightly different sign conventions for the differential on C_f, which give isomorphic but not quite identical complexes.