TCC Homological Algebra: Assignment #3

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This is the third of 4 problem sheets. Solutions should be submitted to me (via any appropriate method) by **noon on 30th November**. This problem sheet will be marked out of a total of 25; the number of marks available for each question is indicated.

Note that rings are assumed to be unital (i.e. having a multiplicative identity element 1), ring homomorphisms are assumed to map 1 to 1, and modules are left modules, unless otherwise stated.

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1. [2 points] Let $\mathcal{C}$ be an abelian category with enough injectives, and X an object of $\mathcal{C}$. Show that X is injective if and only if it is F-acyclic for every left-exact functor F from $\mathcal{C}$ to an abelian category.

2. [2 points] Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of abelian groups, and $I_X^\bullet, I_Y^\bullet, I_Z^\bullet$ injective resolutions of X, Y, Z. Is it necessarily the case that the maps in the short exact sequence lift to a short exact sequence of complexes $0 \to I_X^\bullet \to I_Y^\bullet \to I_Z^\bullet \to 0$? Give a proof or counterexample as appropriate.

3. Let $S : 0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ be a short exact sequence in an abelian category $\mathcal{C}$. We say the sequence S is *split* if there is an isomorphism $Y \xrightarrow{\cong} X \oplus Z$ compatible with the maps $X \to Y$ and $Y \to Z$.

   (a) [1 point] Show that the following are equivalent:

      (i) The exact sequence S is split.

      (ii) There exists a morphism $\pi : Y \to X$ such that $\pi \circ \alpha = \mathrm{id}_X$.

      (iii) There exists a morphism $\phi : Z \to Y$ such that $\beta \circ \phi = \mathrm{id}_Z$.

   (b) [1 point] Show that if either X is injective, or Z is projective, then S must be split.

   (c) [2 points] Assume $\mathcal{C}$ is R-Mod for some ring R. Prove the following converse to (b): if X is such that *every* short exact sequence starting with X is split, then X is an injective object.

4. [3 points] Let k be a field, and let R be the ring $k[X,Y]/(X^2, XY, Y^2)$. Let $I = (X, Y)$ be the unique maximal ideal of R, so that $R/I \cong k$. Find a projective resolution of R/I as an R-module, and hence compute the groups $\mathrm{Ext}^i_R(R/I, R)$. *(Hint: Do not expect your projective resolution to have only finitely many terms!)*

5. [2 points] Let R be a commutative ring, and A, B, C any three R-modules, with A projective. Show that
$$\mathrm{Ext}^i_R(A \otimes_R B, C) = \mathrm{Hom}_R(A, \mathrm{Ext}^i_R(B, C))$$
for every $i \geq 1$. *(You may assume the statement is true for $i = 0$).*

6. Let G be a group. Recall the "bar resolution" $X_n(G)$ of the trivial $\mathbf{Z}[G]$-module $\mathbf{Z}$, discussed in lectures.

   (a) [2 points] Write down the differential $d_1 : X_2(G) \to X_1(G)$ explicitly. Use this and the formula for d_0 given in lectures to show that for any G-module M we have $H^1(G, M) = Z^1(G, M)/B^1(G, M)$ where
$$Z^1(G, M) = \{\text{functions } \sigma : G \to M \text{ such that } \sigma(gh) = \sigma(g) + g\sigma(h)\}$$

and

$$B^1(G,M) = \{\text{functions such that } \sigma(g) = gm - m \text{ for some } m \in M\}.$$

(b) [2 points] Let M be a G-module. An *extension of* $\mathbf{Z}$ *by* M is a short exact sequence of G-modules

$$0 \longrightarrow M \longrightarrow E \longrightarrow \mathbf{Z} \longrightarrow 0,$$

for some E; two such extensions are equivalent if there is a morphism between the two short exact sequences which is the identity on M and on $\mathbf{Z}$. Show that there is a bijection between $H^1(G,M)$ and the equivalence classes of extensions of $\mathbf{Z}$ by M.

7. [2 points] Let G be a group and M a $k[G]$-module, where k is a field of characteristic 0. Show that for any normal subgroup $H \trianglelefteq G$ of finite index, the restriction map

$$\mathrm{res}_{G/H} : H^1(G,M) \to H^1(H,M)^{G/H}$$

is an isomorphism.

8. [3 points] Let $G = C_2 = \{1, \sigma\}$.
   (a) Show that

   $$\ldots \mathbf{Z}[G] \xrightarrow{\sigma-1} \mathbf{Z}[G] \xrightarrow{\sigma+1} \mathbf{Z}[G] \xrightarrow{\sigma-1} \mathbf{Z}[G]$$

   is a projective resolution of the trivial module, as a $\mathbf{Z}[G]$-module.
   (b) Hence compute the cohomology groups of
      i. $\mathbf{Z}$ with the trivial G-action;
      ii. $\mathbf{Z}$ with the generator σ acting as -1.

9. [3 points] Let G be the "infinite dihedral group", which has two generators a, b satisfying $b^2 = 1$ and $bab = a^{-1}$. Let H be the normal subgroup generated by a. Write down the E_2 sheet of the Hochschild–Serre spectral sequence for the cohomology of the trivial G-module $\mathbf{Z}$ in terms of cohomology of H and G/H, and hence show that the groups $H^k(G,\mathbf{Z})$ (for trivial G-action on $\mathbf{Z}$) are $\mathbf{Z}$ for $k = 0$, trivial for k odd, and have order 4 for k even.

   (You may assume that the generator of $G/H \cong C_2$ acts on $H^1(H,\mathbf{Z})$ as -1.)

10. [Not assessed] Find an pair of δ-functors $S = (S^n)^{n\geq 0}$ and $T = (T^n)^{n\geq 0}$ such that $S^0 = T^0$ but S is not isomorphic to T.