TCC Homological Algebra: Assignment #4

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This is the last of 4 problem sheets. Solutions should be submitted to me (via any appropriate method) by **noon on Monday 9th January**. This problem sheet will be marked out of a total of 25; the number of marks available for each question is indicated.

Note that rings are assumed to be unital (i.e. having a multiplicative identity element 1), ring homomorphisms are assumed to map 1 to 1, and modules are left modules, unless otherwise stated. Calligraphic letters $\mathcal{C}$ and $\mathcal{D}$ refer to arbitrary abelian categories.

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1. [1 point] A *first-quadrant homological spectral sequence in $\mathcal{C}$* is exactly the same data as a first-quadrant cohomological spectral sequence in $\mathcal{C}^{\mathrm{op}}$. Write out carefully a formulation of this definition in terms of objects and morphisms in $\mathcal{C}$ (i.e. without mentioning $\mathcal{C}^{\mathrm{op}}$).

2. (a) [2 points] Let R be a commutative ring. For $n \geq 0$, the Tor functor $\mathrm{Tor}_n^R : R\text{-}\underline{\mathrm{Mod}} \times R\text{-}\underline{\mathrm{Mod}} \to R\text{-}\underline{\mathrm{Mod}}$ is defined by

$$\mathrm{Tor}_n^R(A, B) = L_n(- \otimes_R B)(A) = L_n(A \otimes_R -)(B).$$

Show that if $A_\bullet$ is a chain complex of projective R-modules, with $A_i = 0$ for $i < 0$, and B is any R-module, then we have the following:

   i. a first-quadrant homological spectral sequence in R-$\underline{\mathrm{Mod}}$

$$E^2_{pq} = \mathrm{Tor}_p^R(H_q(A_\bullet), B) \Rightarrow H_q(A_\bullet \otimes_R B).$$

   ii. a first-quadrant cohomological spectral sequence in R-$\underline{\mathrm{Mod}}$

$$E_2^{pq} = \mathrm{Ext}_R^p(H_q(A_\bullet), B) \Rightarrow H^q(\mathrm{Hom}_R(A_\bullet, B)).$$

   (b) [2 points] Show that if $R = \mathbf{Z}$ we have short exact sequences

$$0 \to H_n(A_\bullet) \otimes B \to H_n(A_\bullet \otimes B) \to \mathrm{Tor}_1^{\mathbf{Z}}(H_{n-1}(A_\bullet), B) \to 0$$

and

$$0 \to \mathrm{Ext}^1_{\mathbf{Z}}(H_{n-1}(A_\bullet), B) \to H^n(\mathrm{Hom}(A_\bullet, B)) \to \mathrm{Hom}(H_n(A_\bullet), B) \to 0$$

for every $n \geq 0$.

3. Let $X^\bullet$ be an object of $\mathrm{Ch}^\bullet(\mathcal{C})$. We say $X^\bullet$ is *split exact* if all cohomology objects $H^n(X^\bullet)$ are zero and the short exact sequences

$$0 \longrightarrow Z^n(X^\bullet) \longrightarrow X^n \xrightarrow{d^n} B^{n+1}(X^\bullet) \longrightarrow 0$$

are split for every n.

   (a) [2 points] Show that the identity map $\mathrm{id}_{X^\bullet}$ is null-homotopic if and only if $X^\bullet$ is split exact.

   (b) [2 points] Suppose $X^\bullet$ is an injective object in $\mathrm{Ch}^\bullet(\mathcal{C})$. Show that $X^\bullet$ is a split exact complex of injective objects of $\mathcal{C}$. *[Hint: To show that injective $\Rightarrow$ split exact, consider the injective map of complexes $X^\bullet \to$* $\mathrm{cone}(\mathrm{id}_{X^\bullet})$*. By assumption the identity map $X^\bullet \to X^\bullet$ must extend to* $\mathrm{cone}(\mathrm{id}_X)$*. What does this imply?]*

(c) [1 point] Prove the converse: split exact sequences of injective objects of $\mathcal{C}$ are injective objects of $\mathrm{Ch}^\bullet(\mathcal{C})$.

4. [1 point] Let $\underline{\mathrm{Vect}}(k)$ denote the category of vector spaces over some field k. Show that every cochain complex over $\underline{\mathrm{Vect}}(k)$ is quasi-isomorphic to a complex with all differentials zero.

5. [3 points] Let R be a left-Noetherian ring, and let $X^\bullet$ be a bounded-above cochain complex of R-modules, such that $H^i(X)$ is finitely-generated as an R-module for all i.

   (a) Show that there exists a subcomplex $Y^\bullet$ of $X^\bullet$ such that every Y^i is finitely-generated as an R-module, and the inclusion $Y^\bullet \hookrightarrow X^\bullet$ is a quasi-isomorphism.

   (b) Show that if there exists $M \in \mathbf{Z}$ such that $H^i(X)$ is zero for $i < M$, there exists a bounded cochain complex of finitely-generated modules $Z^\bullet$ and a quasi-isomorphism $Y^\bullet \to Z^\bullet$.

6. Let E_r^{pq} be a first-quadrant cohomological spectral sequence in $\underline{\mathrm{Ab}}$ (starting at some $r = r_0$). Suppose that $E_{r_0}^{pq}$ is a finite group for all p, q, and there is some N such that $E_{r_0}^{pq}$ is zero when $p + q > N$.

   (a) [1 point] Show that for all $r > r_0$, E_r^{pq} is finite for all p, q, and is zero if $p + q > N$.

   (b) [3 points] Show that the product

$$\prod_{\substack{p,q \geq 0 \\ p+q \leq N}} \left(\#E_r^{pq}\right)^{(-1)^{p+q}}$$

   is independent of $r \geq r_0$.

   (c) [1 point] Show that in the above setting, if E_r^{pq} converges to some limit $(X^n)^{n \geq 0}$, then X^n is finite for all n and $X^n = 0$ for $n > N$, and the product

$$\prod_{n=0}^{N} (\#X^n)^{(-1)^n}$$

   is equal to the common value of the products from part (b).

   (d) [1 point] Hence show that if G is the group $\mathbf{Z}^m$, for any $m \geq 1$, then for every finite $\mathbf{Z}[G]$-module M, the cohomology groups $H^i(G, M)$ are finite for all i and zero for $i > m$, and we have

$$\prod_{i=0}^{m} \left(\#H^i(G, M)\right)^{(-1)^i} = 1.$$

7. [2 points] Let $X \xrightarrow{f} Y$ be a morphism in $\mathrm{Ch}^\bullet(\mathcal{C})$, and C_f its mapping cone (see sheet 2). Let α_f and β_f denote the natural maps $Y \to C_f$ and $C_f \to X[1]$, so there is a triangle

$$X \xrightarrow{f} Y \xrightarrow{\alpha_f} C_f \xrightarrow{\beta_f} X[1].$$

   If D denotes the mapping cone of α_f, construct an isomorphism of triangles in $K(\mathcal{C})$ between the triangles

$$Y \xrightarrow{\alpha_f} C_f \xrightarrow{\alpha_{\alpha_f}} D \xrightarrow{\beta_{\alpha_f}} Y[1]$$

   and

$$Y \xrightarrow{\alpha_f} C_f \xrightarrow{\beta_f} X[1] \xrightarrow{f[1]} Y[1].$$

   (This is one of the axioms for $K(\mathcal{C})$ being a triangulated category.)

8. [3 points] Let $I^\bullet$ and $X^\bullet$ be two cochain complexes over $\mathcal{C}$, supported in degrees ≥ 0 (that is, $I^p = X^p = 0$ for all $p < 0$), with I^p an injective object for all p. Let $f^\bullet : I^\bullet \to X^\bullet$ be a quasi-isomorphism.

   Show that there is a cochain map $g^\bullet : X^\bullet \to I^\bullet$ such that $g \circ f$ is homotopic to $\mathrm{id}_{I^\bullet}$.