# MA4H9 Modular Forms: Problem Sheet 3

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*This is the third of three problem sheets, each of which amounts to 5% of your final mark for the course. This problem sheet will be marked out of a total of 40; the number of marks available for each question is indicated. You should hand in your work to the Undergraduate Office by **3pm on Monday 10th January 2011**.*

Throughout this sheet, $\Gamma$ is a finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and $k \in \mathbb{Z}$ (some questions will make additional assumptions on $k$).

1. [6 points] Let $f$ be a nonzero modular function of level $\Gamma$ and weight $k$. Recall that the total valence $V_\Gamma(f)$ was defined by
$$V_\Gamma(f) = \sum_{c \in C(\Gamma)} v_{\Gamma,c}(f) + \sum_{z \in \Gamma \backslash \mathcal{H}} \frac{v_z(f)}{n_\Gamma(z)}.$$
Give a careful proof of Lemma 2.4.5, which states that for $g \in \mathrm{SL}_2(\mathbb{Z})$ the total valence of $f$ and $f \mid_k g$ are related by
$$V_{g^{-1}\Gamma g}(f \mid_k g) = V_\Gamma(f).$$

2. [3 points] In my first research paper, I found myself needing the following identity of weight 0 modular functions of level $\Gamma_0(2)$:
$$\frac{E_6^2}{\Delta} = \frac{(1 + 2^6 f_2)(1 - 2^9 f_2)^2}{f_2}$$
where $f_2(z) = \frac{\Delta(2z)}{\Delta(z)}$. I verified by a computer calculation that the $q$-expansions of both sides agreed up to $q^N$, for some sufficiently large $N$. How large an $N$ did I need to use? *(Hint: Clear denominators and apply Corollary 2.4.7).*

3. [2 points] Show that commensurability is an equivalence relation on the set of subgroups of a fixed group $G$.

4. [3 points] Let $k \geq 1$ and let $N_k(\Gamma)$ be the Eisenstein subspace of $M_k(\Gamma)$ (defined as the orthogonal complement of $S_k(\Gamma)$ with respect to the Petersson product). Show that $N_k(\Gamma)$ is preserved by the action of $[\Gamma g \Gamma]$ for any $g \in \mathrm{GL}_2^+(\mathbb{Q})$.

5. [3 points] Let $f$ be the unique normalised eigenform in $S_2(\Gamma_0(11))$, and let $g = f^2$. Calculate the first two terms of the $q$-expansions of $g$ and of $T_2(g)$, and hence show that $\dim S_4(\Gamma_0(11)) \geq 2$.

6. [6 points] Let $N \geq 2$ and let $\chi$ be a Dirichlet character mod $N$. Let $a \in \mathbb{Z}/N\mathbb{Z}$ and define the *Gauss sum*
$$\tau(a, \chi) = \sum_{b \in (\mathbb{Z}/N\mathbb{Z})^\times} e^{2\pi i ab/N} \chi(b).$$
   (a) Show that $\tau(a,\chi) = \overline{\chi(a)}\tau(1,\chi)$ if $a \in (\mathbb{Z}/N\mathbb{Z})^\times$.
   (b) Let $M \mid N$. Show that if $\chi$ does not factor through $(\mathbb{Z}/M\mathbb{Z})^\times$, then we have
$$\sum_{\substack{b \in (\mathbb{Z}/N\mathbb{Z})^\times \\ b = 1 \bmod M}} \chi(b) = 0.$$
   Hence show that if $\chi$ is primitive, $\tau(a,\chi) = 0$ for $a \neq (\mathbb{Z}/N\mathbb{Z})^\times$.

(c) Calculate $\tau(1,\chi)$ when $N = p^j$ ($p$ prime, $j \geq 1$) and $\chi$ is the trivial character mod $p^j$.

7. [4 points] Let $N \geq 2$ and let $H$ be a subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$. Define $\widehat{H}$ to be the subgroup of Dirichlet characters $\chi$ mod $N$ such that $\chi(d) = 1$ for all $d \in H$.

   (a) Show that $\Gamma_H(N) = \left\{ \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N) : a, d \in H \right\}$ is a finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

   (b) Show that for any $k \geq 1$ we have
   $$S_k(\Gamma_H(N)) = \bigoplus_{\chi \in \widehat{H}} S_k(\Gamma_1(N), \chi).$$

8. [4 points] Suppose $p$ is a prime, $\Gamma = \Gamma_1(p)$ and $g = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Find $p$ matrices $(g_j)_{j=0,\ldots,p-1}$ in $\mathrm{GL}_2^+(\mathbb{Q})$ such that
   $$\Gamma g \Gamma = \bigsqcup_{0 \leq j < p} \Gamma g_j = \bigsqcup_{0 \leq j < p} g_j \Gamma.$$

9. [5 points] Let $V$ be a finite-dimensional complex vector space endowed with a positive definite inner product (a finite-dimensional Hilbert space). Let $A : V \to V$ be a linear operator.

   (a) Show that if $A$ is selfadjoint, $\langle Ax, x \rangle$ is real for all $x \in V$.

   (b) We say $A$ is *positive semidefinite* if it is selfadjoint and $\langle Ax, x \rangle \geq 0$ for all $x \in V$. Show that if $A$ is positive semidefinite, there is a unique positive semidefinite $B$ such that $B^2 = A$. (We write $B = \sqrt{A}$.)

   (c) Show that for any linear operator $A$, the operator $A^*A$ is positive semidefinite, and if $P = \sqrt{A^*A}$, then we may write $A = UP$ with $U$ unitary. Show conversely that if $A = UP$ with $U$ unitary and $P$ positive semidefinite, we must have $P = \sqrt{A^*A}$. Is $U$ uniquely determined?

   (d) Show that $A$ is normal if and only if we can find a unitary $U$ and positive semidefinite $P$ such that $A = UP$ and $U$ and $P$ commute.

   (e) Find an example of a nondegenerate (but not positive definite!) inner product space $V$ and a linear operator $A : V \to V$ which is normal but not diagonalisable.

10. [4 points] Let $f$ be weakly modular of level $\Gamma$ and weight $k$. Let $f^* : \mathcal{H} \to \mathbb{C}$ be the function defined by $f^*(z) = \overline{f(-\overline{z})}$. Show that $f^*$ is weakly modular of weight $k$ and level $\Gamma^* = \sigma^{-1}\Gamma\sigma$, where $\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

    Show that $f^*$ is a modular function, modular form, or cusp form if and only if $f$ is, and that the $q$-expansions of $f^*$ and $f$ at $\infty$ are related by $a_n(f^*) = \overline{a_n(f)}$.

11. *[Non-assessed and for amusement only]* Let $M(\Gamma) = \bigoplus_{k \geq 0} M_k(\Gamma)$, which is clearly a ring. Show that for any $\Gamma$, $M(\Gamma)$ is finitely generated as an algebra over $\mathbb{C}$, and we may take the generators to have weight at most 12.