Recall V irred smooth $GL_2(F)$ rep^n, not 1-dim^l

- $V^N = 0$
- $\mathrm{Hom}_N(V, \Theta)$ 1-dim^l $\forall\, \Theta$ nontriv char of N
- If $v \in V$, and $\forall\, \Theta$ char of N

and $\psi \in \mathrm{Hom}_N(V, \Theta)$ have $\psi(v) = 0$, then $v = 0$.

Now fix any one nontriv^l Θ.
Every other char of N is of form $x \mapsto \Theta(ax)$, some $a \in F$.

<u>Def^n</u> Fix λ basis of $\mathrm{Hom}_N(V, \Theta)$
For $v \in V$, let ϕ_v = fcn on $F^\times$ def by
$\phi_v(a) = \lambda\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v\right)$ (Kirillov fcn of v)

$K(V, \Theta)$ = space of fcns on $F^\times$ consisting of the ϕ_v (Kirillov model of V)

Think of this as a "Fourier expansion coefficient" of V.

<u>Prop</u>
(i) If $v \in V$, $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B$, then

$$\phi_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)v}(x) = \omega_V(d)\, \Theta\left(\frac{bx}{d}\right) \phi_v(ax/d)$$

ω_V = central char of V.
(ii) $v \mapsto \phi_v$ is an inj^n.
(iii) Any $\phi \in K(V)$ is supported on $F^\times \cap$ (compact set in F)
(iv) $K(V) \supseteq C_c^\infty(F^\times)$
(v) $K(V)/C_c^\infty(F^\times)$ is finite-dim^l, and is 0 iff V is supercuspidal.

(Note (i) & (iv) $\Rightarrow$ all SC rep^{n}s are isomorphic as reps of $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$)

<u>Pf</u> (i) is immediate from def^{n}s.
For (ii): if $\phi_v = 0$, then $\psi(v) = 0$ $\forall$ homs ψ to non-triv^l chars of N.
Let $w = nv - v$, $n \in N$. This now dies in <u>every</u> N character quotient $\Rightarrow w = 0$ by Lemma (i)
$\Rightarrow v \in V^N = 0$.
(iii): since V smooth, $v = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} v$ for some $b \neq 0$
so $\phi_v(x) = \Theta(bx)\, \phi_v(x)$ $\forall x$.
For $v(x) \ll 0$, $\Theta(bx) \neq 1$ so $\phi_v(x)$ must be 0.
(v) Show first that $K(V)/K(V) \cap C_c^\infty$ is finite-dim^l.
Pf: consider $V(N)$ = subspace gen by $nv - v$ $\forall v \in V$, $n \in N$
$V/V(N) = V_N$ finite-dim^l (& 0 if V SC)
Hence $V(N) \neq 0$.
We compute that $\phi_{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)v - v}(x) = \underbrace{\left[\Theta(bx) - 1\right]}_{0 \text{ if } v(x) \gg 0} \phi_v(x)$
So $\phi_v \in C_c^\infty$ for $v \in V(N)$.
(iv): since $V(N) \neq 0$, $K(V) \cap C_c^\infty(F^\times) \neq 0$
but $C_c^\infty(F^\times)$ is irred as a B-rep^n. □

§4.3 Pf of Casselman's Thm

Prop For $n \in \mathbb{Z}$ let $\overline{N}_n = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} : x \in \mathfrak{p}^n \right\}$

Then $\overline{N}_n$ and $U_\infty = \begin{pmatrix} \mathcal{O}^\times & 0 \\ 0 & 1 \end{pmatrix}$ (← zero!) generate U_n for $n \geq 0$, & if $n \leq -1$ they generate a gp containing $SL_2(F)$.

Pf Exercise.

Now let V as in theorem, $v \mapsto \phi_v$ Kirillov model

Easy check: $v \in V^{U_\infty} \iff \phi_v$ supported on $\mathcal{O}$ & const. on cosets of $\mathcal{O}^\times$

From prop: $v \in V^{U_n}$ iff $v \in V^{U_\infty}$ and $v \in V^{\overline{N}_n}$ $(n \geq 0)$, and

if $v \in V^{U_\infty} \cap V^{\overline{N}_{-1}}$, then $v = 0$.

Key lemma $\begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}$ maps V^{U_n} to a subspace of $V^{U_{n+1}}$ of $\boxed{\text{codim} \leq 1}$

Pf If $v \in V^{U_{n+1}}$ and $\phi_v(1) = 0$ then ϕ_v is identically 0 on $\mathcal{O}^\times$. Hence $\begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}^{-1} v$ corresp. to a fcn on $F^\times$ supported in $\mathcal{O}$ + stable under scaling by $\mathcal{O}^\times \Rightarrow$ preserved by U_∞ & also stable under $\overline{N}_n$

thus $\begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}^{-1} v \in V^{U_n}$.

So subspace of $V^{U_{n+1}}$ st $\phi_v(1) = 0$ is in image.

Conclusion of pf Let c = least n st $V^{U_n} \neq 0$.

Then Key Lemma $\Rightarrow V^{U_c}$ is 1-dim^l, and if v basis, then $\phi_v(1) \neq 0 \Rightarrow$ codim in Key Lemma is 1 $\forall n \geq c$, and v basis of $V^{U_{n+1}}$ / image of V^{U_n} by $\begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}$

□

Remarks

- Qty $c = c(V)$ is conductor of V

  Can show
$$c(V) = \begin{cases} c(\chi) + c(\psi), & V = I(\chi, \psi) \\ \max(1, 2c(\gamma)) & V = St \otimes \gamma \\ \geq 2 & V \text{ supercuspidal} \end{cases}$$

- Can compute iso class of V from Hecke ops on V^{U_c} [L-Weinstein, 2012]

CHAPTER 5 ADÈLE GROUPS

§5.1 Adeles + Ideles of $\mathbb{Q}$

Def^n $\mathbb{A}_f = \prod'_{\ell \text{ prime}} \mathbb{Q}_\ell$

$= \{ (x_\ell) \in \prod_\ell \mathbb{Q}_\ell : x_\ell \in \mathbb{Z}_\ell \text{ for most } \ell \}$

("most" = all but finitely many)

$\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ (but won't use this much)

Topology on $\mathbb{A}_f$: prods $\prod_v U_v$, $U_v \subset \mathbb{Q}_v$ open & $U_v = \mathbb{Z}_v$ for most v.

$\Rightarrow \hat{\mathbb{Z}} = \{ (x_\ell) : x_\ell \in \mathbb{Z}_\ell \ \forall \ell \}$ is open in $\mathbb{A}_f$ + profinite

Thus $(\mathbb{A}_f, +)$ is a loc prof. gp

Easy fact: $\mathbb{Q}$ is dense in $\mathbb{A}_f$ (by Chinese remainder thm)

Def^n $\mathbb{A}_f^\times$, finite ideles, with topology given by inclusion $\mathbb{A}_f^\times \hookrightarrow \mathbb{A}_f \times \mathbb{A}_f$, $x \longmapsto (x, x^{-1})$

(Fun exercise: this is really different from subspace top from $\mathbb{A}_f$!)

This makes $\mathbb{A}_f^\times$ a loc prof gp

Fact: $\mathbb{Q}^\times$ is discrete in $\mathbb{A}_f^\times$ (as $\mathbb{Q}^\times \cap \hat{\mathbb{Z}}^\times = \pm 1$ is finite)

Prop For any U open cpct in $\mathbb{A}_f^\times$, $\mathbb{A}_f^\times / \mathbb{Q}_{>0}^\times U$ is finite. If $U = \hat{\mathbb{Z}}^\times$ it is trivial.

Proof STP the 2nd statement.
After some unravelling this amounts to: for any collection of integers n_p, p prime, most $= 0$, $\exists\, x \in \mathbb{Q}_{>0}^\times$ st $v_p(x) = n_p \ \forall p$ (let $x = \prod p^{n_p}$)

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§5.2 Numbers Fields

$F/\mathbb{Q}$ finite ext^n

$\mathbb{A}_{F,f} = \prod'_{v \text{ prime of } F} F_v$ $\mathbb{A}_F$, $\mathbb{A}_{F,f}^\times$, etc similar

- F dense in $\mathbb{A}_{F,f}$
- $F^\times$ not dense in $\mathbb{A}_{F,f}^\times$, but not discrete either, as $F^\times \cap \hat{\mathcal{O}}_F^\times = \mathcal{O}_F^\times$ may not be finite
- for any $U \leq \mathbb{A}_{F,f}^\times$ open cpct, $\mathbb{A}_{F,f}^\times / F_{>0}^\times U$ finite, but there can be no U for which it's triv^l

$\# \mathbb{A}_{F,f}^\times / F_{\gg 0}^\times \hat{\mathcal{O}}_F^\times$ is narrow class number

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§5.3 GL_2 and SL_2

Prop $\forall N \geq 1$, the reduction map

$$SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/N)$$

is surjective.

Pf Everyone's seen this. $\square$

Similarly $SL_2(\mathcal{O}_F) \twoheadrightarrow SL_2(\mathcal{O}_F/\mathfrak{n})$, $\mathfrak{n} \triangleleft \mathcal{O}_F$ ideal

+ also SL_n. But analogue for GL_2 or GL_1 is false.

Corollary $SL_2(\mathbb{Q})$ is dense in $SL_2(\mathbb{A}_f)$.

Pf Closure of $SL_2(\mathbb{Q})$ contains $SL_2(\hat{\mathbb{Z}})$ by prop^n. But Cartan decomp^{n}s for $SL_2(\mathbb{Q}_\ell)$

$$\Rightarrow SL_2(\mathbb{A}_f) = \coprod_{m \geq 1} SL_2(\hat{\mathbb{Z}}) \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} SL_2(\hat{\mathbb{Z}})$$

so closure of $SL_2 \mathbb{Q}$ is everything.

(Corollary also true for $SL_2(F)$ in $SL_2(\mathbb{A}_{F,f})$ – but not via this proof!)

Analogue for GL_2 doesn't work, of course.

– – – – –

$\mathcal{H}$ = upper half-plane in $\mathbb{C}$, $\circlearrowleft GL_2^+(\mathbb{R})$

Let $\Gamma \subset SL_2(\mathbb{Z})$ congruence subgp ($\Leftrightarrow$ closure $U = \bar{\Gamma}$ in $SL_2(\hat{\mathbb{Z}})$ is open, + $\Gamma = U \cap SL_2(\mathbb{Q})$).

Prop

$$\underbrace{SL_2(\mathbb{Q})}_{\text{diag}^l \text{ left action}} \backslash SL_2(\mathbb{A}_f) \times \mathcal{H} / \underbrace{U}_{\text{right action on } SL_2(\mathbb{A}_f)}$$

is canonically $\Gamma \backslash \mathcal{H}$, via $\tau \mapsto (1, \tau)$

Pf Given any $(g, \tau) \in SL_2(\mathbb{A}_f) \times \mathcal{H}$, $\exists \gamma$ st $\gamma g \in U$, by density. Thus

$$(g, \tau) \sim (\gamma g, \gamma \tau) \sim (1, \gamma \tau)$$

OTOH, if $(1, \tau) \sim (1, \tau')$, then $\exists \gamma \in SL_2(\mathbb{Q})$ st $(1, \tau') = (\gamma u, \gamma \tau)$ $(u \in U)$

$$\Rightarrow \gamma \in U \cap SL_2(\mathbb{Q}) = \Gamma$$

Now consider GL_2. Here strong approximation fails, so adelic objects give something new.

Let $U \subset GL_2(\mathbb{A}_f)$ open cpct. $\Gamma = U \cap GL_2^+(\mathbb{Q})$ doesn't uniquely determine U.

Eg. $U = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}$, $\left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \bmod N \right\}$, $\left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}$

all have $U \cap GL_2^+(\mathbb{Q}) = \Gamma_1(N)$.

Thm Let $U \subset GL_2(\mathbb{A}_f)$ open cpct. Then

$$Y(U) = GL_2^+(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) \times \mathcal{H} / U$$

is a manifold with finitely many connected components, each non-canonically isomorphic to a quotient of $\mathcal{H}$.

More precisely: let $g_1 \ldots g_n$ be any set of elts of $GL_2(\mathbb{A}_f)$ whose determinants are reps of

$$\mathbb{A}_f^\times / \mathbb{Q}_{>0}^\times \cdot \det(U)$$

Then $\Gamma_i = GL_2^+(\mathbb{Q}) \cap g_i U g_i^{-1}$ is a subgp of $SL_2(\mathbb{Q})$ commensurable with $SL_2(\mathbb{Z})$, and

$$\coprod_{i=1}^{n} \Gamma_i \backslash \mathcal{H} \xrightarrow{\sim} Y(U)$$

$$\tau \text{ in } i^{\text{th}} \text{ component} \longmapsto (g_i, \tau)$$