$\phi_f(x) =$ coeff of $e^{2\pi i \tau}$ in $f\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \tau\right)$

$f \in M_{k,t}$, $k \in \mathbb{Z}, t \in \mathbb{R}$

**Prop**

(i) $\phi_f$ supported on $\mathbb{A}_f^\times \cap$ (cpct set in $\mathbb{A}_f$)

(ii) for $n \in \mathbb{Q}^{\times +}$, $\phi_f(nx)$
$$= n^{-t} \left(\text{coeff of } e^{2\pi i n \tau} \text{ in } f\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \tau\right)\right)$$

(iii) $\phi_{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} f}(x) = \Theta(bx)\, \phi_f(ax)$

$\Theta : \mathbb{A}_f \to \mathbb{C}^\times$ unique smooth char st
$\Theta(x) = e^{-2\pi i x}$, $x \in \mathbb{Q}$ $\left( \mathbb{A}_f / \hat{\mathbb{Z}} = \mathbb{Q}/\mathbb{Z} \right)$

(iv) $f \mapsto \phi_f$ is injective

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**Proof** (i) is very similar to local version

(ii) Look at how $\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$ acts

(iii) formal if $bx \in \mathbb{Q}$ + follows $\forall b$ via strong approx

(iv) Clear from (ii) that $\phi_f$ determines $f\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \tau\right)$ $\forall x, \tau$. But $\begin{pmatrix} \mathbb{A}_f^\times & 0 \\ 0 & 1 \end{pmatrix}$ contains a set of reps for $GL_2^+(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) / U$ for any open $U$. $\square$

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Now let $\Pi$ be an irred subrep$^n$ of $S_{k,t}$
(a "cuspidal automorphic rep$^n$" of $GL_2(\mathbb{A}_f)$ of wt $k,t$ )

<u>Prop</u> i) Have $\Pi = \bigotimes_\ell' \Pi_\ell$ where $\Pi_\ell$ is infinite-dim$^l$ irred rep of $GL_2(\mathbb{Q}_\ell)$

ii) The space $K(\pi) = \{ \phi_f : f \in \Pi \}$
then $K(\pi) \cong \bigotimes' K(\pi_\ell, \Theta_\ell)$
via the map sending $\phi_{\otimes x_\ell}$ to $\bigotimes_\ell \phi_{x_\ell}$ □

<u>Corollary</u> Let $\Pi_1, \Pi_2$ be two CAR's of wt $(k,t)$
st $\Pi_1 \cong \Pi_2$ as $GL_2(\mathbb{A}_f)$ rep$^n$s.
Then $\Pi_1 = \Pi_2$.

<u>Pf</u> Uniqueness of Kirillov model: if $\Pi_{1,\ell} \cong \Pi_{2,\ell}$
then $K(\pi_{1,\ell}, \Theta_\ell)$ and $K(\pi_{2,\ell}, \Theta_\ell)$ are the same function spaces on $\mathbb{Q}_\ell^\times$
If true $\forall \ell \Rightarrow K(\pi_1) = K(\pi_2)$
But can recover the fcn $f$ from $\phi_f$
$\Rightarrow \pi_1 = \pi_2$ as function spaces inside $S_{k,t}$

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<u>Thm</u> (Strong multiplicity one)
Let $\pi_1, \pi_2$ be CARs of same wt as before
Suppose $\Pi_{1,\ell} \cong \Pi_{2,\ell}$ for most $\ell$
Then $\Pi_1 = \Pi_2$

<u>Pf</u> Choose a finite set of primes $S$ containing all $\ell$ st $\Pi_{1,\ell} \ncong \Pi_{2,\ell}$

For $\ell \in S$ let $\phi_{\ell,1} = \mathbb{1}_{\mathbb{Z}_\ell^\times} \in K(\pi_{1,\ell}, \Theta_\ell)$
For $\ell \notin S$ choose any $\phi_\ell \in K(\pi_{1,\ell}) = K(\pi_{2,\ell})$
most equal to the spherical vector
Then $\phi = \bigotimes_\ell \phi_\ell \in K(\pi_1) \cap K(\pi_2)$
Since $\Pi_1$ and $\Pi_2$ are irred, and
$S_{k,t} \hookrightarrow$ (fcns on $\mathbb{A}_f^\times$),
this shows $\Pi_1 \cap \Pi_2 \neq 0 \Rightarrow \Pi_1 = \Pi_2$ □

<u>Remarks</u>
(i) Ramakrishnan has shown that if $\Pi_1 \neq \Pi_2$,
$\{ \ell : \Pi_{1,\ell} \cong \Pi_{2,\ell} \}$ has density $\leq 7/8$

(ii) Analogue of mult. 1 for $SL_2$ is true
but naive analogue of strong mult one is <u>false</u>

## § 7.3 Concrete consequences of mult. one, etc

**Prop** $S_{k,t} = \bigoplus_{\pi \text{ CAR wt }(k,t)} \pi$

**Pf** By twisting can assume $t = k/2$.
$S_{k,k/2}$ is unitarizable $\Rightarrow$ direct sum of irreducibles. □

This also shows summands are orthogonal.

For each $\pi$, let $c(\pi) = \prod_\ell \ell^{c(\pi_\ell)}$

By Casselman's thm,
$$\pi^{U_1(N)} = \begin{cases} 0 \text{ if } c(\pi) \nmid N \\ \#\text{ of divisors of } N/c(\pi), \text{ o'wise} \end{cases}$$

In particular, for $N = c(\pi)$, $\pi^{U_1(N)}$ 1-dim'l

**Prop** This gives a bijection
$$\begin{pmatrix}\text{CARs wt} \\ k,t\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{normalized new} \\ \text{eigenforms in } S_k(\Gamma_1(N)) \\ \text{some } N\end{pmatrix}$$
$\pi \longleftrightarrow$ mod form spanning $\pi^{U_1(N)}$, $N = c(\pi)$

**Pf** Given $\pi$ CAR, let $f_\pi$ = any gen of $\pi^{U_1(c(\pi))}$

Then $f_\pi(1, -)$ is a mod form of level $N = c(\pi)$

$f_\pi$ is an eigenvector for $T_\ell$, $U_\ell$ operators because these are double cosets $\begin{bmatrix}\varpi_\ell & 0 \\ 0 & 1\end{bmatrix}$ up to scaling

WLOG $a_1(f_\pi) = 1$

$f_\pi$ is new because $\pi$ is orthogonal to all CARs of conductor $< c(\pi)$ □

Upshot: $S_k(\Gamma_1(N))$ decomposes as
$$\bigoplus_{\substack{\pi \text{ CAR} \\ \text{of cond} \\ c(\pi) \mid N}} \underbrace{S_k(\Gamma_1(N))[\pi]}_{\text{spanned by } f_\pi(d\tau),\ d \mid N/c(\pi)}$$

+ if $f_1, f_2$ two normalized newforms, $a_\ell(f_1) \neq a_\ell(f_2)$ for $\infty$ly many prime $\ell$.

Note that if $f_\pi$ new vect of $\pi$, $\pi = \bigotimes'_\ell \pi_\ell$, then $\forall \ell \nmid$ (level of $f_\pi$), we have $\pi_\ell \cong I(\alpha_\ell, \beta_\ell)$

$\alpha_\ell, \beta_\ell$ unram chars sending $\ell$ to roots of
$$X^2 - \frac{a_\ell(f_\pi)}{\ell^{t-\frac{1}{2}}} X + \ell^{k-2t} \varepsilon(\ell),$$
$\varepsilon$ = character of $f_\pi$

---

## Twisting

Dirichlet chars $\left(= \text{homs } (\mathbb{Z}/N)^\times \to \mathbb{C}^\times \text{ some } N\right)$

$\updownarrow$

$\left(\text{smooth chars of } \mathbb{A}_f^\times / \mathbb{Q}^{\times +}\right)$

If $\chi$ such a character, $\pi$ aut. rep., then can form $\pi \otimes (\chi \circ \det)$

**Prop** $\pi \otimes \chi$ is automorphic (of same wt as $\pi$)

**Pf** let $f \in \pi$. Consider

$$(g,\tau) \mapsto \chi(\det g) f(g,\tau)$$

This is clearly in $S_{k,t}$ + generates a repⁿ iso to $\pi \otimes \chi$. □

**But** this fcn evaluated at $(1,\tau)$ is just $f$!

$f \in S_k(\Gamma_1(N))$ extends uniquely to $f \in (S_{k,t})^{U_1(N)}$

but there are lots of other subgps of $GL_2(\mathbb{A}_f)$ extending $\Gamma_1(N)$!

**Prop** If $\pi$ CAR, then the space of fcns on $\mathcal{H}$, $\{f(1,\tau) \mid f \in \pi\}$, is spanned by all the $GL_2^+(\mathbb{Q})$ translates of $f_{\pi \otimes \chi}$, $\chi$ Dir char.

**Pf** WLOG $f \in \pi$ invᵗ under $\left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N\right\}$ some $N$

Decompose $\pi^{\left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N\right\}}$ as a repⁿ of $\begin{pmatrix} \hat{\mathbb{Z}}^\times & 0 \\ 0 & 1 \end{pmatrix}$

$\chi$-eigenspace gives fcns whose restrictions are $GL_2^+(\mathbb{Q})$ translates of $f_{\pi \otimes \chi}$.

## 7.4 Hilbert mod forms

"Everything works as before, when you set it up right"

Choose $F$ tot real, $\sigma_1 \cdots \sigma_d : F \hookrightarrow \mathbb{R}$

$(\underline{k}, \underline{t}) \in \mathbb{Z}^d \times \mathbb{R}^d$ st $k_i - 2t_i = m$ indep of $i$

We defined $S_{(\underline{k},\underline{t})} \circlearrowleft GL_2(\mathbb{A}_{F,f})$ earlier

Each $f \in S_{(\underline{k},\underline{t})}$, restricted to $g \times \mathcal{H}^d$ some $g$, is a fcn of $(\tau_1 \cdots \tau_d) \in \mathcal{H}^d$ inv$^t$ under translation by some lattice $L \subset F$ (commensurable with $\mathcal{O}_F$) acting via $x \cdot (\tau_1 \cdots \tau_d)$

$= (\tau_1 + \sigma_1(x), \cdots, \tau_d + \sigma_d(x))$

Hence $\exists$ Fourier exp$^n$ of form

$$f(g, \underline{\tau}) = \sum_{\substack{y \in F \\ (F^\times \text{ if } f \text{ cuspidal})}} a_y \, e^{2\pi i(\tau_1 \sigma_1(y) + \cdots + \tau_d \sigma_d(y))}$$

where $e^{2\pi i(\cdots)} = \underline{q}^y$

Def$^n$ For $f \in S_{(\underline{k},\underline{t})}$, set

$\phi_f$ = fcn on $\mathbb{A}_{F,f}^\times$

$\phi_f(x)$ = coeff of $\underline{q}^1$ in $f(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \tau)$

As before, can use this to prove a strong mult 1 thm.

By-product: "Fourier-Whittaker exp$^n$"
for any $f \in S_{\underline{k},\underline{t}}$, $\exists$ smooth fcn $\phi_f$ on $\mathbb{A}_{F,f}^\times$ st

$$f(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \underline{\tau}) = \sum_{\substack{\xi \in F^\times \\ \xi \gg 0}} \xi^{\underline{t}} \, \phi_f(\xi x) \, \underline{q}^{\xi}$$

$[\xi^{\underline{t}} = \prod \sigma_i(\xi)^{t_i} \in \mathbb{R}]$

If $f$ is $U_1(\mathfrak{n})$ ($\mathfrak{n}$ ideal of $\mathcal{O}_F$) then $\phi_f$ is $\hat{\mathcal{O}}_F^\times$-inv$^t$ so it is morally a fcn on fractional ideals: $\phi_f(I) = \phi_f(\text{any gen of } I\hat{\mathcal{O}}_F)$

One has $\phi_f(I) = 0$ unless
$I = \partial_F^{-1} J$, $J$ integral,
($\partial_F$ different)

If normalize st $\phi_f(\partial_F^{-1}) = 1$,
then $\phi_f(\partial_F^{-1} \mathfrak{p}) = \dfrac{T_{\mathfrak{p}}\text{-eigenvalue of } f}{Nm(\mathfrak{p})}$

for eigenforms $f$.

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Twisting is more complicated

$\hat{\mathcal{O}}_F^\times \longrightarrow \mathbb{A}_{F,f}^\times / F^{\times +}$ not an iso.

Concretely, this means most fcns of form $f(1, -)$, $f \in S_{(\underline{k},\underline{t})}$ are not inv$^t$ under $\begin{pmatrix} \mathcal{O}_F^{\times +} & 0 \\ 0 & 1 \end{pmatrix}$ hence not in span of newforms

# Chapter 8 Eisenstein Series

$\mathcal{E}_{k,t}$ = orth. complement of $S_{k,t}$ in $M_{k,t}$

## §8.1 Reminders

$\Gamma \subset SL_2(\mathbb{Q})$ commensurable with $SL_2\mathbb{Z}$

Cusps $C(\Gamma)$ = $\Gamma$-orbits on $\mathbb{P}^1(\mathbb{Q})$ (finite set)

Say $c \in C(\Gamma)$ is irregular if $Stab_\Gamma(c)$ contains an elt conjugate to $\begin{pmatrix} -1 & * \\ 0 & -1 \end{pmatrix}$ (some $*$)

else all elts conj to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ (regular)

If $-1 \in \Gamma$, all cusps irregular!

Let $\mathcal{E}_k(\Gamma) = \{ f \in M_k(\Gamma) \mid \langle f, g \rangle = 0 \ \forall g \in S_k(\Gamma) \}$

Any $f \in \mathcal{E}_k(\Gamma)$ has values $f(c) \in \mathbb{C}$ for $c \in C(\Gamma)$

$\mathcal{E}_k(\Gamma) \hookrightarrow \mathbb{C}\, C(\Gamma)$

Fact If $k$ is odd, $f(c) = 0$ $\forall$ irregular cusps,

so $\mathcal{E}_k(\Gamma) \hookrightarrow \mathbb{C}\, C_{reg}(\Gamma)$

Prop (i) If $k \geq 3$, then $\mathcal{E}_k(\Gamma) \longrightarrow \begin{cases} \mathbb{C}\, C(\Gamma) \\ \text{resp } \mathbb{C}\, C_{reg}(\Gamma) \end{cases}$ is a bijection

(ii) If $k = 2$, cokernel is 1-dim$^l$, with unique relation being

$$\sum_c \text{width}(c)\, f(c) = 0 \quad \forall f \in \mathcal{E}_2(\Gamma)$$

Pf Explicit series computations $\square$

## §8.2 $\mathcal{E}_{k,t}$ as a rep$^n$ of $GL_2$

Def$^n$ For $f \in \mathcal{E}_{k,t}$, let

$\alpha(f) = f(1, \infty)$

($f(1, \tau)$ is a mod form of some level, thus has a value at cusp $\infty$)

Prop i) if $a, d \in \mathbb{Q}^\times$, $ad > 0$, then

$$\alpha\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} f\right) = d^k (ad)^{-t} \alpha(f)$$

ii) if $x \in \mathbb{A}_f$, $\alpha\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} f\right) = \alpha(f)$