Chapter 10 L-fcns of mod forms

10.1 The standard L-fcn

$f = \sum a_n q^n$ normalized newform
level N, char. ε, wt k

$L(f,s) = \sum_{n \geq 1} a_n n^{-s}$ cvgt for $\operatorname{Re} s > \begin{cases} 1 + k/2 & \text{(easy)} \\ \frac{1+k}{2} & \text{(hard)} \end{cases}$

Euler product

$$L(f,s) = \prod_{\ell \text{ prime}} \left(1 - a_\ell \ell^{-s} + \varepsilon(\ell) \ell^{k-1-2s}\right)^{-1}$$

interpret $\varepsilon(\ell) = 0$ if $\ell \mid N$

Thm (Hecke): if we let
$L_\infty(f,s) = (2\pi)^{-s} \Gamma(s)$
$\Lambda(f,s) = L_\infty(f,s) L(f,s)$
then (a) $\Lambda(f,s)$ has analytic cont^n to all $s \in \mathbb{C}$
(b) functional eq^n:

$$\Lambda(f, k-s) = N^{s-k/2} \varepsilon(f) \overline{\Lambda(f, \bar{s})}$$
$$= \qquad \Lambda(f^*, s)$$

$f^* = \sum \bar{a}_n q^n$ (scalar, abs value 1)

(see D & S §5.10 or Miyake §4.3)

Note: local factor at ℓ
knows T_ℓ and $\langle \ell \rangle$
$\leftrightarrow$ determined by π_ℓ, π CAR gen by f

Notation $\pi \subset S_{k,t}$ gen. by f
$L(\pi, s) = L(f, s + t - \tfrac{1}{2})$
func^l eqn now relates $L(\pi, s)$ and $L(\pi^\vee, 1-s)$

§10.2 Rankin–Selberg

f, g newforms $\left(\begin{array}{l}\text{levels } N_f, N_g \\ \text{wts } k_f, k_g, \ldots\end{array}\right)$

$\sum_{n \geq 1} a_n(f) a_n(g) n^{-s}$ – will have Euler prod

factor at prime $\ell \nmid N_f N_g$

$$\frac{\left(1 - \varepsilon_f(\ell)\varepsilon_g(\ell)\ell^{k_f+k_g-2-2s}\right)}{\left(1-\alpha_f\alpha_g\ell^{-s}\right)\left(1-\alpha_f\beta_g\ell^{-s}\right)\left(1-\beta_f\alpha_g\ell^{-s}\right)\left(1-\beta_f\beta_g\ell^{-s}\right)}$$

α_f, β_f roots of Hecke poly of f (& g sim)

Naive ("imprimitive") R-S L-fcn

$$L^{imp}(f, g, s) = L^{(N)}(\varepsilon_f\varepsilon_g, 2s+2-k_f-k_g) \sum_{n \geq 1} a_n(f) a_n(g) n^{-s}$$

discard terms for $\ell \mid N_f N_g$

Mysterious thing: This sometimes has fund^l eqn $s \leftrightarrow k_f + k_g - s$ but not always: sometimes need to modify by polys in ℓ^{-s} for $\ell \mid N_f N_g$

Thm (Jacquet)

(a) If π_1, π_2 irred ∞-dim^l reps of $GL_2(F)$, F nonarch local,
$\exists$ fcn $L(\pi_1 \times \pi_2, s) = \dfrac{1}{(\text{poly in } q^{-s})}$
+ if π_1, π_2 are spherical, it is

$$\left[\left(1-\alpha_1\alpha_2 q^{-s}\right)\left(1-\alpha_1\beta_2 q^{-s}\right)\left(1-\beta_1\alpha_2 q^{-s}\right)\left(1-\beta_1\beta_2 q^{-s}\right)\right]^{-1}$$

α_1, β_1 values at ϖ_F of chars from which π_1 induced

(b) Π_1, Π_2 C.A.R.'s of $GL_2(\mathbb{A}_F)$

$$L(\Pi_1, \Pi_2, s) = \prod_\ell L(\pi_{1,\ell}, \pi_{2,\ell}, s)$$

has meromorphic cont^n & a fund^l eqn

Goal today: Sketch pf.

10.3 Local theory

F nonarch local fld

π_1, π_2 irred ∞-dim^l $GL_2(F)$ reps

ω_2, ω_2 cent. chars, $\omega = \omega_1\omega_2$.

$K(\pi_i)$ Kirillov spaces.

<u>Def^n</u> for $v_1 \in \pi_1$, $v_2 \in \pi_2$, $s \in \mathbb{C}$,

$$Z(s, v_1, v_2) := \int_{F^\times} \phi_{v_1}(-x)\, \phi_{v_2}(x)\, |x|^{s-1}\, d^\times x \quad (\text{HM on } F^\times)$$

$$\tilde{Z}(\qquad) := \int \quad \cdot \quad \omega^{-1}(x)\, d^\times x$$

<u>Prop</u> (i) Both converge for $Re(s) \gg 0$ (dep. on π_i) + have mero. cont^n as rat^l fns of q^{-s}; poles bounded indep. of v_i.

(ii) $Z\left(s, \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} v_1, \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} v_2\right) = |a/d|^{1-s}\, \omega(d)\, Z(s, v_1, v_2)$

$\tilde{Z}$ $\qquad\qquad \cdots \quad \omega(a) \quad \cdots$

Eqv^{t}ly: $\exists$ nonzero G-homs, well-def. for most s,

$$Z : \pi_1 \otimes \pi_2 \longrightarrow I(|\cdot|^{\frac{1}{2}-s}, |\cdot|^{s-\frac{1}{2}}\omega)$$

$$\tilde{Z} \qquad \longrightarrow I(|\cdot|^{\frac{1}{2}-s}\omega, |\cdot|^{s-\frac{1}{2}})$$

(expresses a branching law for restricting $GL_2 \times GL_2$ reps to diag^l GL_2.)

Want to pair against suitable elts of PS rep^{n}s.

<u>Def^n</u> for $\Phi \in C_c^\infty(F^2)$, $g \in GL_2(F)$, $s \in \mathbb{C}$

$$f_\Phi(g) = |\det g|^s \int_{F^\times} \Phi((0,t)g)\, \omega(t)\, |t|^{2s}\, d^\times t$$

$$\tilde{f}_\Phi(g) = |\det g|^s\, \omega(\det g)^{-1} \int \cdots \quad \omega^{-1}(t)\, |t|^{2s}\, d^\times t$$

<u>Prop</u> (i) For fixed g, Φ, f and $\tilde{f}$ cvgt $Re(s) \gg 0$, & have mero. cont^n w. bounded poles.

(ii) When defined, $f_\Phi \in I(|\cdot|^{s-\frac{1}{2}}, |\cdot|^{\frac{1}{2}-s}\omega^{-1})$

$\tilde{f}_\Phi \in I(|\cdot|^{s-\frac{1}{2}}\omega^{-1}, |\cdot|^{\frac{1}{2}-s})$

<u>Def^n</u> $\Psi(s, v_1, v_2, \Phi) = \langle Z(s, v_1, v_2), f_\Phi \rangle$

$\tilde{\Psi} \qquad = \quad \cdots$

(Jacquet's zeta integrals)

<u>Thm</u> (a) For fixed v_1, v_2, Φ these are rat^l fns of q^{-s}.

(b) $\exists!$ poly $Q(X)$ st $Q(0) = 1$ and

$\Xi(s, v_1, v_2, \Phi) = Q(q^{-s})\, \Psi(\cdots)$

$\in \mathbb{C}[q^s, q^{-s}]$ & the Ξ for varying v_1, v_2, Φ generate unit ideal of $\mathbb{C}[q^s, q^{-s}]$.

(c) If π_i unram then $Q(q^{-s})$ is local L-factor stated above, & for v_1, v_2, Φ spherical vectors, $\Xi(\cdots) = 1$

(d) Local fun^l eq^n:

$$\tilde{\Xi}(1-s, v_1, v_2, \hat{\Phi}) = \omega_1(-1)\, \varepsilon(s, \pi_1, \pi_2)\, \Xi(s, v_1, v_2, \Phi)$$

where $\hat{\Phi}(x,y) := \iint \Phi(u,v)\, \theta(xv - yu)\, du\, dv$.

$\varepsilon(s, \pi_1, \pi_2)$ fcn of form Aq^{Bs}.

<u>Define</u> $L(\pi_1 \times \pi_2, s) = Q(q^{-s})$

Follows that $\tilde{Q}(q^{-s}) = L(\pi_1^\vee \times \pi_2^\vee, s)$

§10.6 Global theory

π_1, π_2 CAR's wts $(k_i, \frac{k_i}{2})$

Last § $\Rightarrow$ local factors $L(\pi_{1,\ell}, \pi_{2,\ell}, s)$

WTS $\prod_\ell$ (these) has anal. cont. + fc^l eqn.

Def^n $\Phi \in C_c^\infty(\mathbb{A}_f^2)$, $\tau \in \mathcal{H}$, $s \in \mathbb{C}$ st $Re(s) > 1$

$$E_\Phi^{(k)}(\tau, s) = \frac{\Gamma(s+k/2)}{\pi^{s-k/2}(-2\pi i)^k} \sum_{\substack{(m,n)\in \mathbb{Q}^2\\ \neq (0,0)}} \frac{\Phi(m,n)\, \mathrm{Im}(\tau)^{s-k/2}}{(m\tau+n)^k |m\tau+n|^{2s-k}}$$

For fixed s, Φ, transforms like a mod form under $GL_2^+(\mathbb{Q}) \cap \mathrm{Stab}(\Phi)$

Define $E_\Phi^{(k)}(g, \tau, s) =$

$$\|\det g\|^s \int_{\mathbb{A}_f^\times/\mathbb{Q}^{\times +}} E^{(k)}_{\left(\begin{smallmatrix} a & 0\\ 0 & a\end{smallmatrix}\right) g\Phi}(\tau, s)\, \omega(a) \|a\|^{2s}\, d^\times(a)$$

$$\tilde{E} = \|\det g\|^s \omega(\det g)^{-1} \int \dots \omega^{-1}(a) \dots$$

E and $\tilde{E}$ transform like elts of $M_{k, k/2}$

Thm $\tilde{E}_{\hat{\Phi}}^{(k)}(1-s) = E_\Phi^{(k)}(s)$

Def^n For $f_1 \in \pi_1$, $f_2 \in \pi_2$, $\Phi \in C_c^\infty$, $[k_1 \geq k_2$ WLOG$]$

set $\Psi(s, f_1, f_2, \Phi)$

$$= \langle f_1^*, f_2 E_\Phi^{(k_1 - k_2)}(-, s) \rangle$$

$\left(f_1^*(g, \tau) = \overline{f_1\left(\left(\begin{smallmatrix} -1 & 0\\ 0 & 1\end{smallmatrix}\right) g, -\bar{\tau}\right)}\right)$

$\tilde{\Psi}$ similarly, using $\tilde{E}$.

Easy (-ish) for f_i, Φ fixed these are holo in s + have mero cont^n to all $s \in \mathbb{C}$, only poles if $k_1 = k_2$ and $s = 0$ or 1

Also

$$\tilde{\Psi}(1-s, v_1, v_2, \hat{\Phi}) = \Psi(s, v_1, v_2, \Phi)$$

<u>Thm</u> ("Unfolding" the global integral)

if $f_1 = \bigotimes_\ell' f_{1,\ell}$,

$\Phi = \bigotimes_\ell' \Phi_\ell$, then

$$\Psi(s, f_1, f_2, \Phi) = \left[\prod_\ell \Psi(s, f_{1,\ell}, f_{2,\ell}, \Phi_\ell)\right] \times L_\infty(s)$$

+ same with $\sim$.

(assuming $\mathrm{Re}(s) \gg 0$ so product converges)

Since $\Psi(s, f_{1,\ell}, f_{2,\ell}, \Phi_\ell) = L(\pi_{1,\ell} \times \pi_{2,\ell}, s) \times \Xi(\ldots)$

& Ξs at good primes are 1,

we have

$$\Psi(s, f_1, f_2, \Phi) = L(\pi_1 \times \pi_2, s) L_\infty(s) \times \left(\prod_\ell \Xi(s, f_{1,\ell}, \ldots)\right)$$

finite product

$\Rightarrow$ mero cont^n of $L(\pi_1 \times \pi_2, s) L_\infty(\ldots)$

Can <u>choose</u> local test data so Ξ term nonzero at given s $\Rightarrow$ analytic cont^n if $k_1 \neq k_2$

Func^{l} eq^{n}

$$\Psi(s, f_1, f_2, \Phi) = \Lambda(\pi_1 \times \pi_2, s)\left(\prod_{\ell} \Xi(s, f_{1,\ell}, \dots)\right)$$

$$\|$$

$$\widetilde{\Psi}(1-s, f_1, f_2, \hat{\Phi}) = \Lambda(\pi_1^\vee \times \pi_2^\vee, 1-s)\left(\prod_{\ell} \widetilde{\Xi}(1-s, f_{1,\ell}, f_{2,\ell}, \hat{\Phi}_\ell)\right)$$

Know $\widetilde{\Xi}(1-s, \dots) = \varepsilon(\pi_{1,\ell} \times \pi_{2,\ell}, s)\, \Xi(s, \dots)$

+ deduce

$$\boxed{\Lambda(\pi_1 \times \pi_2, s) = \varepsilon(\pi_1 \times \pi_2, s)\, \Lambda(\pi_1^\vee \times \pi_2^\vee, 1-s)}$$

Remark If $\pi_1, \pi_2 \longleftrightarrow$ newforms f_1, f_2

have Galois reps

$$\rho_{f_1,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\overline{\mathbb{Q}}_p)$$

for any p

$$L(\pi_{1,\ell} \times \pi_{2,\ell}, s) = \det\left(1 - \ell^{-s}\mathrm{Frob}_\ell^{-1} \mid (\rho_{f_1,p} \otimes \rho_{f_2,p})^{I_\ell}\right)^{-1}$$

for any $\ell \neq p$

Imprimitive L-factor sees only

$$(\rho_{f_1,p})^{I_\ell} \otimes (\rho_{f_2,p})^{I_\ell}$$