AG for NT

1 Sheaves of Modules

Let X be a topological space. Recall what a sheaf is: $F : U \mapsto F(U)$, $V \subset U$ then we have a map $F(U) \to F(V)$ with a uniqueness and existence property

We have $(X, \mathcal{O}_X)$ is a scheme.

Definition. A sheaf F of abelian groups on X is an $\mathcal{O}_X$*-module* if each $F(U)$ is an $\mathcal{O}_X(U)$-module in such a way that for $V \subset U$, $s \in F(U)$ and $t \in \mathcal{O}_X(U)$ we have $(t \cdot s)|_V = t|_V \cdot s|_V \in \mathcal{O}_X(V)$.

A *morphism of* $\mathcal{O}_X$*-modules* $F \to G$ is a morphism of sheafs $F \to G$ such that each $F(U) \to G(U)$ is a $\mathcal{O}_X(U)$-module homomorphism.

Remark. Each F_x is an $\mathcal{O}_X$-module.

Example. New from Old:

1. $(f_i)_{i \in I}$, $\mathcal{O}_X$ -module then the sheaf associated to $u \mapsto \oplus_{i \in I} f_i(U)$ is also an $\mathcal{O}_X$-module, $\oplus_{i \in I} f_i$.
2. If F, G are $\mathcal{O}_X$-modules then sheaf associated to$u \mapsto F(U) \otimes_{\mathcal{O}_X} G(U)$ is an $\mathcal{O}_X$-module. Denoted $F \otimes_{\mathcal{O}_X} G$.
3. $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, if F is an $\mathcal{O}_X$-module then $f_* F$ is an $f_* \mathcal{O}_X$-module. Have $f_\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$. So $f_* F$ becomes an $\mathcal{O}_Y$-module.
4. As above, G an $\mathcal{O}_Y$-module, then $F^{-1} G$ is an $f^{-1} \mathcal{O}_Y$-module. We have $f_\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ induces $f^{-1} \mathcal{O}_Y \to \mathcal{O}_X$. So $\mathcal{O}_X$ is also an $f^{-1} \mathcal{O}_Y$-module. Define $f^* G := f^{-1} G \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{O}_X$. This is an $\mathcal{O}_X$-module

Definition. A sheaf F of $\mathcal{O}_X$-module is *locally free* if we can cover X by open subset U_i such that $F | U_i$ is isomorphic to a direct sum of copies of $\mathcal{O}_X|_{U_i}$. And if we can just take one copy, we say that F is an *invertible sheaf.*

Example (Key Example)**.** e Let A be a ring, M an A-module, $X = \operatorname{Spec} A$. We will define an $\mathcal{O}_X$-module $\widetilde{M}$ as follows. For $f \in A$, set $\widetilde{M}(D(f)) = M_f \cong M \otimes_A A_f$, and $\mathcal{O}_X(D(f)) \cong A_f$ so M_f is an $\mathcal{O}_X(D(f))$-module. The restrictions maps $M_f \to M_g$ for $D(g) \subset D(f)$ is given by $\otimes_A M$ the map $A_f \to A_g$.

Exercise. Show that $\widetilde{M}$ is a $\mathcal{B}$-sheaf, where $\mathcal{B} = \{D(f) | f \in A\}$.

Extend to $\widetilde{M}$ sheaf on X which is an $\mathcal{O}_X$-module

What are the stalks: Let $f \in \operatorname{Spec} A$, $(\widetilde{M})_f \cong \varinjlim_{D(f) \ni f} M_f \cong \varinjlim_{D(f) \ni f} A_f \otimes_A M \cong M \otimes_A \varinjlim_{D(f) \ni f} A_f \cong M \otimes_A A_f \cong M_f$

Remark. Given $M \to N$ an A-module homomorphism, we get $\mathcal{O}_X$-module morphism $\widetilde{M} \to \widetilde{N}$ by localizing.

Conversely, $\widetilde{M} \to \widetilde{N}$ induces an A-module homomorphism $\widetilde{M}(X) = M \to \widetilde{N}(X) = N$. (This is done by taking global sections)

Lemma 1.1. *Let* $X = \operatorname{Spec} A$*. Then*

1. $\{M_i\}_{i \in I}$ *a collection of* A*-module, then* $\widetilde{\oplus_{i \in I} M_i} \cong \oplus_{i \in I} \widetilde{M_i}$

2. *$L \to M \to N$ of A-modules is exact if and only if $\widetilde{L} \to \widetilde{M} \to \widetilde{N}$ is exact. (i.e., exact on stalks)*

3. *$\widetilde{M \otimes_A N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$*

4. *Let $\phi : A \to B$ be a ring homomorphism. This induces $f : \operatorname{Spec} B \to \operatorname{Spec} A$. Let M be a B-module. Then $f_*\widetilde{M} \cong \widetilde{M}$ where the second $\widetilde{M}$ is viewed as an A-module via ϕ. Let N be an A-module then $f^*(\widetilde{N}) \cong \widetilde{N \otimes_A B}$.*

5. *Let $f \in A$. $(D(f), \mathcal{O}_{\operatorname{Spec} A}|_{D(f)}) \cong \operatorname{Spec} A$ (using the map $A \to A_f$). Let M be an A-module, $\widetilde{M}|_{D(f)} \cong \widetilde{M_f}$ as $\mathcal{O}_{\operatorname{Spec} A_f}$-module.*

Proof. Exercise □

Definition 1.2. Let $(X, \mathcal{O}_X)$ be a scheme. An $\mathcal{O}_X$-module F is *quasi-coherent* if we can cover X by open affine $U_i = \operatorname{Spec} A_i$ such that $F|_{U_i} \cong \widetilde{M_i}$ for some A_i-module M_i.

The sheaf F is *coherent* if we can take each M_i to be finitely generate (as modules)

1.1 Quasi-coherent Sheaves on affine schemes

Proposition 1.3. *If $X = \operatorname{Spec} A$, F a quasi-coherent sheaf on X, then $F \cong \widetilde{M}$ for some A-module M.*

Proof. Observe that: If $F \cong \widetilde{M}$ then $\Gamma(X, F) := F(X)$ is isomorphic to $\Gamma(X, \widetilde{M}) \cong M$. So given any quasi-coherent sheaf F, we will show that $F \cong \widetilde{\Gamma(X, F)}$.

Let $U = D(f)$ be principal open. $F(U)$ is an open $\mathcal{O}_X(U) = A_f$-module. So we have a map $\Gamma(X, F)_f \to F(U)$ defined by $\frac{s}{f^k} \mapsto \frac{s|_U}{f^k}$. This map induces a morphism of sheaves $\widetilde{\Gamma(X, F)} \to F$.

We want to show that this is an isomorphism. So we will show that $\Gamma(X, F)_f \to F(U)$ is an isomorphism for each $f \in A$. This is done using the following lemma

Lemma. *Let $X = \operatorname{Spec} A$. Take $f \in A$, $U = D(f)$, F a quasi-coherent sheaf on X. Then*

1. *If $s \in \Gamma(X, F)$ is such that $s|_U = 0$, ten $\exists n > 0$ such that $f^n s = 0 \in \Gamma(X, F)$*

2. *Given $t \in F(U)$, there is $n > 0$ such that $f^n t$ is the restriction of a $s \in \Gamma(X, F)$ (for some s)*

Remark. 1. gives injectivity and 2. surjectivity of the map is the proposition.

Proof. Part 2. is an exercise

Can cover X by $U_i = \operatorname{Spec} A_i$ such that $F|_{U_i} \cong \widetilde{M_i}$ for some A_i-module M_i. If $D(g) \subset U_i$ then $\widetilde{M_i}|_{D(g)} \cong \widetilde{(M_i)_g}$. So without loss of generality, $U_i = D(g_i)$ for some $g_i \in A$. As $X = \operatorname{Spec} A$ is quasi compact, finitely many g_i will do. $D(f)$ is covered by the sets $D(f) \cap D(g_i) = D(f \cdot g_i)$, and $F(D(f \cdot g_i)) \cong \widetilde{(M_i)_f}$. Let s_i be the image of s in M_i. Then $s_i = 0$ in $(M_i)_f$, so there exists $n > 0$ such that $f^n s_i = 0$ in M_i. By finiteness we can assume n is independent of i. Then $f^n s$ restrict to 0 in each $D(g_i)$. Hence globally $f^n s = 0$. □

□

Proposition 1.4. *Let $X = \operatorname{Spec} A$, F is coherent sheaf on X. If A is Noetherian, then $\Gamma(X, F)$ is finitely generated as an A-module. So in particular $F \cong \widetilde{M}$ for a finitely generated A-module M*

Proof. Exercise □

Corollary 1.5. *Let A be a ring, $X = \operatorname{Spec} A$. Then the function $M \mapsto \widetilde{M}$ gives an equivalence of categories between A-modules and quasi-coherent $\mathcal{O}_X$ modules. The 'Inverse' is $\Gamma(X, -)$.*

If A is Noetherian, same is true fro finitely generated A-modules and coherent $\mathcal{O}_X$-modules.

Corollary 1.6. *If X is a scheme, F an $\mathcal{O}_X$-module, then F is quasi-coherent if and only if every open affine subset $U = \operatorname{Spec} A$, $F|_U \cong \widetilde{M}$ for some A-module M.*

If X is Noetherian, F is coherent, same is true with each M finitely generated.

1.2 Quasi-coherent Sheafs on Proj$S$

Let $S = \oplus_{d\geq 0} S_d$ a graded ring. We have Proj$S$ = {homogeneous prime ideals not containing $S_+ = \oplus_{d>0} S_d$}. Basis $\mathcal{B} = \{D_+(f) | f$ homogeneous, $f \in S_+\}$ (where $D_+(f) = p \in \text{Proj}S | f \notin p\}$)

$\mathcal{O}_X(D_+(f)) \cong S_{(f)} = \{$degree 0 homogenous elements in $S_f\}$. In fact $(D_+(f), \mathcal{O}_X|_{D_+(f)}) \cong \text{Spec}\, S_{(f)}$

Let $M = \oplus_{n\in\mathbb{Z}} M_n$ graded S-module. (So $M_n \subset M_{n+d}$). We want to construct a sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ on X. We do this as follows: Set $\widetilde{M}(D_+(f)) = M_{(f)} = \{$degree 0 homogeneous elements of $M_f\}$. This is an $S_{(f)} = \mathcal{O}_X(D_+(f))$-module. Check that this a $\mathcal{B}$-sheaf for $\mathcal{B} = \{D_+(f)\}$ and check what the restriction maps are.

Set $\widetilde{M}$ to be the resulting sheaf on X. What are the stalks: $(\widetilde{M})_p = M_{(p)}$ =degree 0 homogeneous elements in $M(T^{-1})$ where $T = \{$homogeneous elements not in $p\}$

Fact. *$\widetilde{M}|_{D_+(f)} \cong \widetilde{M_{(f)}}$ is $\mathcal{O}_{\text{Spec}\, S_{(f)}}$-module. In particular $\widetilde{M}$ is quasi-coherent. If S is Noetherian, M is finitely generated, then $\widetilde{M}$ is coherent.*

1.2.1 Twisting

Let S be a graded ring and M a graded S-module, $M = \oplus_{r\in\mathbb{Z}} M_r$. Define $M(n)$ to be the S-module M, but with a different grading given by $M(n)_r = M_{n+r}$. Thus $\widetilde{M(n)}(D_+(f)) = \{$degree n homogenous elements in $M_f\}$.

Definition 1.7. Let S be a graded ring. $X = \text{Proj}S$. For $n \in \mathbb{Z}$, define $\mathcal{O}_X(n)$ to be $\widetilde{S(n)}$. If F is any sheaf of $\mathcal{O}_X$-modules, define $F(n) := F \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Remark. $\mathcal{O}_X(1)$ is called the Twisting Sheaf of Serre.

Twisting is 'well-behaved' provided that S is generated by S_1 as an S_0-algebra. E.g., $A[x_0, \ldots, x_n]$ for some ring A. Indeed, we have the following proposition.

Proposition 1.8. *S is a graded ring, $X = ProjS$. Assume that S is generated by S_1 as an S_0-algebra. Then*

1. *$\mathcal{O}_X(n)$ is an invertible sheaf. (for all n)*
2. *If M is a graded S-module, then $\widetilde{M}(n) \cong \widetilde{M(n)}$.*
3. *$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \cong \mathcal{O}_X(m+n)$*

Proof.

Claim. The set $D_+(f)$ for $f \in S_1$ cover X. Proof is an exercise, uses the assumption S is generated by S_1 as an S_0-algebra.

1. By the claim, it suffices to show that $\mathcal{O}_X(n)|_{D_+(f)}$ is isomorphic to $\widetilde{S_{(f)}}$ as $\mathcal{O}_{\text{Spec}\, S_{(f)}}$-modules. We know that $\mathcal{O}_X(n)|_{D_+(f)} \cong \widetilde{S(n)_{(f)}}$. Suffices to show that $S(n)_{(f)} \cong S_{(f)}$ as $S_{(f)}$-modules. But $S(n)_{(f)}$ =degree n homogeneous elements in S_f. while $S_{(f)}$ =degree 0 homogeneous elements in S_f. We can construct a map $S_f \to S(n)_f$ by $s \mapsto f^n s$. This is an isomorphism as f is invertible in S_f.

2. More generally, we have $\widetilde{M \otimes_S N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ for graded S-modules M, N. <u>But</u> needs the assumption S is generated by S_1 as an S_0-algebra (See Hartshornes for details)

3. Follows from part 2.

□