AG for NT Week 6

Recap

1. A ring, $X = \operatorname{Spec} A$, M an A-module which we associated a sheaf $\widetilde{M}$ of $\mathcal{O}_X$-modules. Have shown: Any quasi-coherent sheafs on $\operatorname{Spec} A$ has this form.

2. S graded ring, $X = \text{Prj}S$, M graded S-module, associated to it $\widetilde{M}$ a quasi-coherent sheaf. $\widetilde{M}(D_+(f)) = \{$homogeneous elements of degree 0 in $M_f\}$.

Want to show: Given <u>any</u> quasi-coherent sheaf on $X = \text{Proj}S$, we have $f \cong \widetilde{M}$ for some graded S-module M. We won't (can't?) say this in general, but

Proposition 0.1. *S graded ring, finitely generated by S_1 as an S_0-algebra ($S = \oplus_{a\geq 0} S_a$). Let $X = \text{Proj}S$ and f a quasi coherent sheaf on X. Then there is a graded S-module M such that $f \cong \widetilde{M}$.*

Remark.

1. Applies to $S = A[x_0, \dots, x_n]$ for some ring A, i.e., to $\mathbb{P}^n_A$

2. Don't have an equivalence of categories between graded S-modules and quasi-coherent sheafs on $\text{Proj}S$. Indeed, 2 different graded modules can give the same sheaf. Example: $\oplus_{n\geq 0} M_n$ and $\oplus_{n \geq n_0} M_n$ for some $n_0 \geq 0$

3. How to define M given f? Cannot just set $M = \Gamma(X, f)$ as in the affine case. Solution is to use twists.

   Example. $S = A[x_0, \dots, x_r]$, $X = \text{Proj}S = \mathbb{P}^r_A$.

   Claim. $\Gamma(X, \mathcal{O}_X(n)) = S_n = \{$homogenous polynomials of degree $n\}$. In particular, $\Gamma(X, \mathcal{O}_X) \cong A$.

   Proof. The sets $D_+(x_i)$ cover X. A section $t \in \Gamma(X, \mathcal{O}_X(n))$ is the same as an $(r+1)$-tuple of sections $(t_0, \dots, t_r)$ with $t_i \in \mathcal{O}_X(n)(D_+(x_i))$ compatible on $D_+(x_i) \cap D_+(x_j) = D_+(x_i x_j)$

   Now, $\mathcal{O}_X(n)(D_+(x_i)) = \{\deg n$ homogeneous elements in $S_{x_i}\}$ and $\mathcal{O}_X(n)(D_+(x_i x_j)) = \{\deg n$ homogeneous elements in $S_{x_i x_j}\}$. We are considering $\mathcal{O}_X(n)(D_+(x_0 \dots x_r)) =$

$\{\deg n$ in homogeneous elements in $S_{x_0\ldots x_r}\}$. We have $\forall i,j,\ S \hookrightarrow S_{x_i} \hookrightarrow S_{x_ix_j} \hookrightarrow S_{x_0\ldots x_r}$ where the maps are defined respectively $f \mapsto f/1$, $f/x_i^k \mapsto fx_j^k/(x_ix_j)^k$ and $f/(x_ix_j)^k \mapsto f(x_0\ldots x_{i-1}x_{i+1}\ldots x_{j-1}x_{j+1}\ldots x_r)^k/(x_0\ldots x_r)^k$. These preserve gradings, injective as no x_i is a zero divisor in S and induce the relevant restriction maps. So to give a section $t \in \Gamma(X, \mathcal{O}_X(n))$ is to give a homogeneous elements of $\deg n$ in $S_{x_0\ldots x_r}$, lying in $\cap_i Sx_i = S$ □

So we have $S \cong \oplus_{n\in\mathbb{Z}}\Gamma(X, \mathcal{O}_X(n)) = \oplus_{n\in\mathbb{Z}}\Gamma(X, \widetilde{S}(n))$.

Definition. S a graded ring, generated by S_1 as S_0-algebra. $X = \text{Proj}S$, f sheaf of $\mathcal{O}_X$-modules. Define the *graded S-module associated to f* to be $\Gamma_*(f) = \oplus_{n\in\mathbb{Z}}\Gamma(X, f(n))$.

Module Structure: If $s \in S_d$, view this as an element of $\Gamma(X, \mathcal{O}_X(d))$ (glue the $S/1$ in $\mathcal{O}_X(d)(D_+(f))$. Then $f(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(d) \cong f(n+d)$. So get homomorphism $\Gamma(X, f(n)) \otimes_{\Gamma(X,\mathcal{O}_X)} \Gamma(X, \mathcal{O}_X(d)) \to \Gamma(X, f(n+d))$ so for $t \in \Gamma(X, f(n))$ define $s \cdot t$ to be the image of $t \otimes s$ in $\Gamma(X, f(n+d))$.

Remark. For $S = A[x_0, \ldots, x_m]$ have shown $S \cong \Gamma_*(\mathcal{O}_X)$ as graded S-modules.

Proposition. *0.1 again. Let S be a graded ring, finitely generated by S_1 as S_0-algebra, $X = ProjS$, f a quasi coherent sheaf on X. Then $f \cong \widetilde{\Gamma_*(f)}$*

Proof. See Hartshorne □

Closed subschemes

Recall: A *closed immersion* is a morphism $f : Y \to X$of schemes such that f induces a homeomorphism from Y to a closed subset of X, and such that $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.

Definition. A *closed subscheme* of a scheme X is an equivalence class of closed immersion where $f : Y \to X$, $f' : Y' \to X$ are equivalent if there is an isomorphism $i : Y' \to Y$ such that $f' = f \cdot i$.

Example. A ring, $a \lhd A$, then homomorphism $A \to A/a$ induces closed immersion $\text{Spec}(A/a) \to \text{Spec}\, A$ image $v(a)$.

Aim: To show that the kernel of $f^\#$, that is, the sheaf $u \mapsto \ker(\mathcal{O}_X(U) \to (f_*\mathcal{O}_Y)(U))$ is quasi-coherent when f is closed immersion.

Definition. A topological space X is *quasi-separated* if the intersection of any two quasi-compact open subsets is again quasi-compact.

Exercise. For any ring A, $\text{Spec}\, A$ is quasi-separated.

Definition 0.2. A morphism of schemes $f : X \to Y$ is *quasi-compact* (respectively *quasi-separated*) if for every affine open $U \supset Y$, $f^{-1}(U)$ is quasi-compact (respectively quasi-seperated)

Remark. If X is a Noetherian scheme, then for any scheme Y and morphism $f : X \to Y$, f is both quasi-compact and quasi-separated.

If $f : X \to Y$ is separated then it's quasi-separated.

Lemma. *Let $i : Y \to X$ be a close immersion. Then i is both quasi-compact and quasi-separated.*

Proof. Let $V = i(Y)$ and take $U \subset X$ affine open (Say $U = \operatorname{Spec} A$). Then $i^{-1}(U)$ is homeomorphic to $U \cap V$. As V is closed in X, $U \cap V$ is closed in U. So $U \cap V$ is of the form $V(a)$ for some ideal $a \lhd A$. Then $i^{-1}(U)$ is homeomorphic to $\operatorname{Spec}(A/a)$ hence both quasi-separated and quasi-compact. □

Lemma. *X a scheme. Then the kernel of any morphism of quasi-compact sheaves $\phi : \mathcal{F} \to \mathcal{G}$ on X is also quasi-coherent.*

Proof. The question is local, so we may assume $X = \operatorname{Spec} A$ is affine. Then there are A-modules, M, N such that $\mathcal{F} \cong \widetilde{M}$ and $\mathcal{G} \cong \widetilde{N}$ and ϕ is induced from a homomorphism $\psi : M \to N$. It is easy to show that $\ker \phi \cong \widetilde{\ker \psi}$. □

Lemma. *Let $f : X \to Y$ be a morphism of schemes. Suppose f is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then $f_*\mathcal{F}$ is a quasi-coherent sheaf on Y.*

Proof. We may assume Y is affine (exercise). Then by assumption, $X = f^{-1}(Y)$ is both quasi-compact and quasi-seperated. So we may cover X by finitely many open affine U_i and for each i, j we may cover $U_i \cap U_j$ by finitely many open affine U_{ijk}.

Now let $V \subset Y$ open. To give a section, $s \in (f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ is the same as giving sections s_i over each $f^{-1}(V) \cap U_i$ agreeing on each $f^{-1}(V) \cap U_{ijk}$. So $f_*\mathcal{F}$ fits into an exact sequence $0 \to f_*\mathcal{F} \to \oplus_i f_*(\mathcal{F}|U_i) \to \oplus_{i,j,k} f_*(\mathcal{F}|U_{ijk})$ of sheaves on Y. As each U_i, U_{ijk} is affine, both $\oplus_i f_*(\mathcal{F}|_{U_i})$ and $\oplus_{i,j,k} f_*(\mathcal{F}|_{U_{ijk}})$ are quasi-coherent (1st lemma last week). Thus, as the kernel of a morphism of quasi-coherent sheaves, $f_*\mathcal{F}$ is also quasi-coherent. □

Corollary. *Let $i : Y \to X$ closed immersion of schemes. Then $i_*\mathcal{O}_Y$ is quasi-coherent.*

Definition. $(X, \mathcal{O}_X)$ be a scheme. A *sheaf of ideals* on X is a sheaf $\mathcal{F}$ on X such that for each $U \subset X$, $\mathcal{F}(U)$ is an ideal of $\mathcal{O}_X(U)$. These are clearly $\mathcal{O}_X$-modules.

Definition. Let Y closed subscheme of X and $i : Y \to X$ corresponding closed immersion. Define the *ideal sheaf of Y, $\mathcal{I}_Y$* to be the kernel of $i^\# : \mathcal{O}_X \to i_*\mathcal{O}_Y$. It depends only on the equivalence class of the closed immersion.

Note. $\mathcal{I}_Y$, as defined above is quasi-coherent.

Example. Let $X = \operatorname{Spec} A$ affine, $Y = \operatorname{Spec}(A/a)$, $i : Y \to X$ obvious closed immersion, then $\Gamma(X, \mathcal{I}_Y) = a$. In particular $\mathcal{I}_Y \cong \widetilde{a}$.

Proposition. *$X = \operatorname{Spec} A$ affine scheme. Then <u>any</u> closed subscheme U of X is of the form $(\operatorname{Spec} A/a, i)$ where $a \lhd A$ and $i : \operatorname{Spec} A/a \to \operatorname{Spec} A$ is induced from $A \to A/a$. This gives a one to one correspondence between ideals of A and closed subschemes of $\operatorname{Spec} A$. In particular, any closed subscheme of an affine scheme is affine.*

Proof. Note that distinct ideals give non-equivalent closed subschemes as can recover a as $\Gamma(X, \mathcal{I}_Y)$ as in example above. Now need a lemma

Lemma. *X a topological space, $i : Y \to X$ continuous and induces homeomorphism from Y to a closed subset of X. Let $\mathcal{F}$ be any sheaf on Y. Then $i^{-1}i_* f$ is canonically isomorphic to $\mathcal{F}$*

Proof. Exercise $\square$

And a definition

Definition. X a topological space, $\mathcal{F}$ a sheaf on X. Define the *support* of $\mathcal{F}$, $\operatorname{supp}(\mathcal{F})$ to be the set $\{x \in X | f_x \neq 0\}$

Let $i : Y \to X = \operatorname{Spec} A$ closed immersion. $\mathcal{I}_Y$ be the corresponding ideal sheaf. Then $\mathcal{I}_Y$ is quasi-coherent so $\mathcal{I}_Y \cong \widetilde{a}$ where $a = \Gamma(X, \mathcal{I}_Y) \lhd A$. Now $i^{\#} : \mathcal{O}_X \to i_*\mathcal{O}_Y$ is surjective, so have a short exact sequence of sheaves. $0 \to \widetilde{a} \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$, but $\mathcal{O}_X/\widetilde{a}$ is isomorphic to $\widetilde{(A/a)}$ canonically, so $i^{\#}$ induces isomorphism $\widetilde{(A/a)} \overset{\sim}{\to} i_*\mathcal{O}_Y$. Now for $p \in \operatorname{Spec} A$ we have

$$(i_*\mathcal{O}_Y)_p \cong \begin{cases} 0 & p \notin i(Y) \\ \mathcal{O}_{Y, i^{-1}(p)} \neq 0 & p \in i(Y) \end{cases}.$$

So $i(Y) = \operatorname{supp}(i_*\mathcal{O}_Y) = \operatorname{Supp}(\widetilde{(A/a)}) = V(a)$. Moreover, have an isomorphism of sheaves of rings on Y, $i^{-1}(\widetilde{A/a}) \overset{\sim}{\to} i^{-1}i_*\mathcal{O}_Y \underset{\text{lemma}}{\overset{\sim}{\to}} \mathcal{O}_Y$ .It is an exercise to use this to define a morphism $Y \to \operatorname{Spec} A/a$ such that

$$\begin{array}{ccc} Y & \xrightarrow{i} & X = \operatorname{Spec} A \\ & \searrow & \uparrow \\ & & \operatorname{Spec} A/a \end{array}$$

commutes

$\square$

Proposition. *Let X be any scheme. Then $Y \mapsto \mathcal{I}_Y$ is a bijection between quasi-coherent sheaves of ideals on X and closed subschemes. If X is Noetherian can replace quasi-coherent by coherent.*

Proof. Let $\mathcal{I}$ be a quasi-coherent sheaf of ideals on X. Set $Y = \mathrm{Supp}(\mathcal{O}_X/\mathcal{I})$ and let $i : Y \to X$ inclusion of topological spaces. Then $i^{-1}(\mathcal{O}_X/\mathcal{I})$ is a sheaf of rings on Y. Need to show that Y is closed in X and that $(Y, i^{-1}(\mathcal{O}_X/\mathcal{I}))$ is a scheme. We can check this locally, so may assume X is affine. Then it follows from the previous proposition. Details are left as an exercise. □

Let S be a graded ring, $I \lhd S$ homogenous ideal. Then the ring homomorphism $S \to S/I$ induces a closed immersion from $\mathrm{Proj}(S/I) \to \mathrm{Proj}(S)$.

Proposition. *Let A be a ring. If Y is a closed subscheme of $\mathbb{P}^r_A$, then there exists a homogenous ideal $I \subset S = A[x_0, \ldots, x_r]$ such that Y is the closed subscheme determined by I.*

Proof. Uses previous results on quasi-coherent sheaves on $\mathbb{P}^r_A$. See Hartshorne for details. □