Sheaves of Modules

X top space

$$\begin{array}{lcl} \mathcal{F} \quad U & \longmapsto & \mathcal{F}(U) \\ \quad\ \ \cup & & \downarrow \\ \quad\ \ V & \longmapsto & \mathcal{F}(V) \end{array}$$

'uniqueness'

'existence'

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$(X, \mathcal{O}_X)$ scheme

<u>Defn</u>: $\mathcal{F}$ sheaf of a.g on X is an $\mathcal{O}_X$-module if each $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module in such a way that for $V \subseteq U$, $s \in \mathcal{F}(U)$, $t \in \mathcal{O}_X(U)$

$$(t \cdot s)|_V = t|_V \cdot s|_V \quad \in \mathcal{O}_X(V)$$

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A <u>morphism of $\mathcal{O}_X$-modules</u> $\mathcal{F} \to \mathcal{G}$ is morphism of sheaves $\mathcal{F} \to \mathcal{G}$ s.t. each $\mathcal{F}(U) \to \mathcal{G}(U)$ is a $\mathcal{O}_X(U)$-module hom.

<u>Remark</u>: Each $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$ module.

'New from old'

- $(\mathcal{F}_i)_{i \in I}$ $\mathcal{O}_X$-modules then sheaf assoc. to

$$U \longmapsto \bigoplus_{i \in I} \mathcal{F}_i(U)$$

is also an $\mathcal{O}_X$-module, $\bigoplus_{i \in I} \mathcal{F}_i$

- $\mathcal{F}, \mathcal{G}$ $\mathcal{O}_X$-modules

Sheaf assoc. to

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$$

is an $\mathcal{O}_X$-mod.

Denoted $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$

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· $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_X)$

If $\mathcal{F}$ is an $\mathcal{O}_X$-module

then $f_* \mathcal{F}$ is an $f_* \mathcal{O}_X$-module

Have $f_\# : \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$.

So $f_* \mathcal{F}$ becomes an $\mathcal{O}_Y$-mod.

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- As above

$\mathcal{G}$ $\mathcal{O}_Y$-mod.

$f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-mod

$f_\#: \mathcal{O}_Y \to f_* \mathcal{O}_X$

induces $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. So $\mathcal{O}_X$ is also $f^{-1}\mathcal{O}_Y$-mod

Define $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$

This is an $\mathcal{O}_X$-mod.

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<u>Def^n</u>: A sheaf F of $\mathcal{O}_X$-modules is <u>locally</u> <u>free</u> if can cover X by open U_i s.t. $F|_{U_i}$ is iso to a direct sum of copies of $\mathcal{O}_X|_{U_i}$

and if can just take
one copy say F is
an invertible sheaf

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Key example

A ring, M an A-module
$X = \operatorname{Spec} A$. Will define an
$\mathcal{O}_X$-module $\widetilde{M}$ as follows.
For $f \in A$. Set $\widetilde{M}(D(f)) = M_f \cong M \otimes_A A_f$
($\mathcal{O}_X(D(f)) \cong A_f$ so M_f an $\mathcal{O}_X(D(f))$-module)

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rest. maps $M_f \to M_g$ $D(g) \subseteq D(f)$

Given by $\otimes_A M$ the map

$A_f \to A_g$.

<u>exerc</u>. Show $\tilde{M}$ is a $\mathcal{B}$-sheaf

Where $\mathcal{B} = \{D(f) \mid f \in A\}$

extend to $\hat{M}$ sheaf on X

which is an $\mathcal{O}_X$-module

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Stalks : $\mathfrak{p} \in \operatorname{Spec} A$

$$(\tilde{M})_{\mathfrak{p}} \cong \varinjlim_{D(f) \ni \mathfrak{p}} M_f$$

$$\cong \varinjlim_{D(f) \ni \mathfrak{p}} A_f \otimes_A M$$

$$\cong M \otimes_A \varinjlim_{D(f) \ni \mathfrak{p}} A_f$$

$$\cong M \otimes_A A_{\mathfrak{p}}$$

$$\cong M_{\mathfrak{p}}$$

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<u>Remark</u>: Given $M \to N$ A-mod hom

get $\mathcal{O}_X$-mod morphism

$$\tilde{M} \to \tilde{N}$$

by localising

Conversely, $\tilde{M} \to \tilde{N}$ induces

A-mod hom $\tilde{M}(X) \to \tilde{N}(X)$

$$\tilde{M}(X) = M, \quad \tilde{N}(X) = N$$

(by taking global sections)

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Lemma: $X = \operatorname{Spec} A$.

Then a) $\{M_i\}_{i \in I}$ A-modules

then $\widetilde{\bigoplus_{i \in I} M_i} \cong \bigoplus_{i \in I} \widetilde{M_i}$

b) $L \to M \to N$ A-modules

is exact $\iff$

$\widetilde{L} \to \widetilde{M} \to \widetilde{N}$ is

(i.e. exact on stalks)

c) $\widetilde{M \otimes_A N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$

d) $\phi : A \to B$ ring hom

$\rightsquigarrow$ $f : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$

Let M a B-module

Then $f_* \widetilde{M} \cong \widetilde{M}$ viewed as an A-mod via ϕ.

M A-module.

Then $f^*(\tilde{M}) \cong \widetilde{M \otimes_A B}$

e) $f \in A$. $(D(f), \mathcal{O}_{\operatorname{Spec} A}|_{D(f)})$

$\cong \operatorname{Spec} A_f \quad (A \to A_f)$

M an A-module.

$\tilde{M}|_{D(f)} \cong \widetilde{M_f}$ as $\mathcal{O}_{\operatorname{Spec} A_f}$-modules

Pf. exerc. $\square$

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Defn: $(X, \mathcal{O}_X)$ - scheme.

An $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if we can cover X by open affine $U_i = \operatorname{Spec} A_i$ i.e. $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for some A_i-modules M_i.

Say $\mathcal{F}$ is coherent if can take each M_i f.g.

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Quasi-Coherent Sheaves on affine Schemes

Prop^n: If $X = \operatorname{Spec} A$, $\mathcal{F}$ q-c sheaf on X, then $\mathcal{F} \cong \tilde{M}$ some A-module M.

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Proof: Observe:

If $\mathcal{F} \cong \widetilde{M}$ then

$\Gamma(X, \mathcal{F}) \cong \Gamma(X, \widetilde{M}) \cong M$
$\mathcal{F}(X)$

So given an q-c $\mathcal{F}$, we will show that $\mathcal{F} \cong \widetilde{\Gamma(X, \mathcal{F})}$

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Let $U = D(f)$ principal open

$\mathcal{F}(U)$ is an $\mathcal{O}_X(U) = A_f$-module. So we have a map

$$\widetilde{\Gamma(X,\mathcal{F})}(U) = \Gamma(X,\mathcal{F})_f \longrightarrow \mathcal{F}(U)$$

$$s/f^k \longmapsto \frac{s|_U}{f^k}$$

This map induces a morphism of sheaves $\widetilde{\Gamma(X,\mathcal{F})} \longrightarrow \mathcal{F}$

RTP isom?.

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STS that

$$\Gamma(X, \mathcal{F})_f \longrightarrow \mathcal{F}(U)$$

is an isom^ for each $f \in A$.

This is done using the following lemma $\square$

Lemma: $X = \operatorname{Spec} A$. $f \in A$, $U = D(f)$, $\mathcal{F}$ a q-c sheaf on X.

Then a) If $s \in \Gamma(X, \mathcal{F})$ is s.t. $s|_U = 0$. Then $\exists\, n > 0$ s.t. $f^n s = 0 \in \Gamma(X, \mathcal{F})$

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b) Given $t \in f(u)$, there is $n > 0$ s.t. $f^n t$ is the restriction of $s \in \Gamma(X, f)$ (for some s)

Rmk: a) is injectivity and b) surjectivity of map in the propn.

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Pf: Can cover X by $U_i = \operatorname{Spec} A_i$ s.t. $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for some A_i-modules M_i.

If $D(g) \subseteq U_i$, then

$$\widetilde{M_i}\big|_{D(g)} \cong \widetilde{M_{i,g}}.$$

So WLOG, $U_i = D(g_i)$ some $g_i \in A$.

As $X = \operatorname{Spec} A$ is q-compact, finitely many g_i will do.

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a) $D(f)$ is covered by the sets $D(f) \cap D(g_i) = D(fg_i)$ and $\mathcal{F}(D(fg_i)) \cong \widetilde{(M_i)_f}$.

Let s_i image of s in M_i.

Then $s_i = 0$ in $(M_i)_f$.

So $\exists\, n > 0$ s.t. $f^n s_i = 0 \in M_i$

(by finiteness n indep of i)

Then $f^n s$ restricts to 0 in each of $D(g_i)$. So $f^n s = 0$ in $\Gamma(X, \mathcal{F})$.

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b) exer. $\square$

<u>Prop^n</u>: $X = \operatorname{Spec} A$, $\mathcal{F}$ coherent sheaf on X. If A is Noetherian, then $\Gamma(X, \mathcal{F})$ is f.g. as an A-module.
So in particular, $\mathcal{F} \cong \widetilde{M}$ for a f.g. A-module M.

<u>Pf.</u> exerc. $\square$

Corollary : A ring, $X = \operatorname{Spec} A$.

Then the functor $M \longmapsto \tilde{M}$ gives an equiv. of categories btw A-modules and q-c $\mathcal{O}_X$-modules.

'Inverse' is $\Gamma(X, -)$.

If A is Noetherian, same is true for f.g. A-modules and coherent $\mathcal{O}_X$-modules.

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<u>Cor</u>: X scheme, $\mathcal{F}$ an $\mathcal{O}_X$-module.
Then $\mathcal{F}$ is q-c iff for
<u>every</u> open affine subset
$U = \operatorname{Spec} A$, $\mathcal{F}|_U \cong \widetilde{M}$
some A-module M.
If X is Noeth, $\mathcal{F}$ coherent,
same is true + M is f.g.

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Quasi-Coherent Sheaves on Proj S

$S = \bigoplus_{d \geq 0} S_d$ graded ring.

$\operatorname{Proj} S = \left\{ \begin{array}{l} \text{homog prime ideals} \\ \text{not containing } S_+ = \bigoplus_{d>0} S_d \end{array} \right\}$

Basis $\mathcal{B} = \left\{ D_+(f) \;\middle|\; \begin{array}{l} f \text{ homog} \\ f \in S_+ \end{array} \right\}$

$D_+(f) = \{ \mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p} \}$

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$$\mathcal{O}_X(D_+(f)) \cong S_{(f)} = \left\{ \begin{array}{l} \text{deg } 0 \text{ homog} \\ \text{elts in } S_f \end{array} \right\}$$

In fact $\left(D_+(f), \mathcal{O}_X|_{D_+(f)}\right) \cong \operatorname{Spec} S_{(f)}$.

$M = \bigoplus_{n \in \mathbb{Z}} M_n$ graded S-module

$(S_d \cdot M_n \subseteq M_{n+d})$

we want to construct
a sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ on X

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Do this as follows.

Set $\tilde{M}(D_+(f)) = M_{(f)} = \left\{ \begin{array}{c} \text{deg } 0 \\ \text{homog elts} \\ \text{of } M_f \end{array} \right\}$

This is an $S_{(f)} = \mathcal{O}_X(D_+(f))$-module.

· Check: This is a $\mathcal{B}$-sheaf for $\mathcal{B} = \{D_+(f)\}$; (restⁿ maps exer.)

Set $\tilde{M}$ to be the resulting sheaf on X.

<u>Stalks</u>: $(\tilde{M})_{\mathfrak{p}} = M_{(\mathfrak{p})}$

$M_{(\mathfrak{p})}$ = deg 0 homog elts in $M[T^{-1}]$ where $T = \{\text{homog elts not in } \mathfrak{p}\}$

<u>Fact</u>: $\tilde{M}|_{D_+(f)} \cong \widetilde{M_{(f)}}$ as $\mathcal{O}_{\operatorname{Spec} S_{(f)}}$-modules.

In particular $\tilde{M}$ is quasi-coherent.

If S Noeth, M f.g. then $\tilde{M}$ is coherent.

Twisting. S graded ring.
M graded S-mod.
$$M = \bigoplus_{r \in \mathbb{Z}} M_r$$

Define $M(n)$ to be the S-module M, but with grading given by
$$M(n)_r = M_{n+r}.$$

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Thus $\widetilde{M(n)}(D_+(f)) = \left\{ \begin{array}{l} \text{deg } n \text{ homog} \\ \text{elts in } M_f \end{array} \right\}$

(different to $\widetilde{M}$).

<u>Def^n</u>: S graded ring, $X = \operatorname{Proj} S$.
For $n \in \mathbb{Z}$ define $\mathcal{O}_X(n)$ to be $\widetilde{S(n)}$. If $\mathcal{F}$ is any sheaf of $\mathcal{O}_X$-modules, define
$$\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n).$$

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Defn: $\mathcal{O}_X(1)$ is called the twisting sheaf of Serre

Twisting is 'well-behaved'
Provided that S is generated
by S_1 as an S_0-algebra.
(e.g. $A[x_0, \dots, x_n]$ Some ring A)
$\rightsquigarrow \mathbb{P}^n_A$

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Indeed, we have

Prop

Prop^n: S graded ring, $X = \text{Proj}\, S$.
Assume S gen by S_1 as an S_0-alg.

Then: a) $\mathcal{O}_X(n)$ is an invertible sheaf. (for all n)
b) If M graded S-module, then $\widetilde{M}(n) \cong \widetilde{M(n)}$

c) $\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \cong \mathcal{O}_X(m+n)$.

Pf: Claim: the sets $D_+(f)$ for $f \in S_1$ cover $X = \operatorname{Proj} S$.

Pf: exer. uses S_1 gen S as an S_0-alg.

a) By claim, STS that $\mathcal{O}_X(n)|_{D_+(f)}$ is isomorphic to $\widetilde{S_{(f)}}$ as $\mathcal{O}_{\operatorname{Spec} S_{(f)}}$-modules.

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We know that $\mathcal{O}_X(n)|_{D_+(f)} \cong \widetilde{S(n)_{(f)}}$.

STS that $S(n)_{(f)} \cong S_{(f)}$ as $S_{(f)}$-modules.

But

$$\begin{array}{ccc} S(n)_{(f)} & \longleftarrow & S_{(f)} \\ \| & & \| \\ \{\text{deg } n \text{ homog elts in } S_f\} & & \{\text{deg } 0 \text{ homog elts in } S_f\} \\ f^n s & \longleftarrow\!\shortmid & s \end{array}$$

This is an iso as f is invertible in S_f.

b) More generally,

$$\widetilde{M \otimes_S N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$$

for graded S-modules M, N.

<u>But</u> needs S gen by S_1 as S_0-alg.

See Hartshorne for details.

c) follows from b)
(exer.)

$\square$

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