Introduction to Hodge Theory and K3 surfaces

Contents

Part I

Hodge Theory (Pierre Py)

Reference: Claire Voisin: *Hodge Theory and Complex Algebraic Geometry*

1 Kähler manifold and Hodge decomposition

1.1 Introduction

Definition 1.1. Let V be a complex vector space of finite dimension, h is a *hermitian form* on V. If $h : V \times V \to \mathbb{C}$ such that

1. It is bilinear over $\mathbb{R}$
2. $\mathbb{C}$-linear with respect to the first argument
3. Anti-$\mathbb{C}$-linear with respect to the second argument
   i.e., $h(\lambda u, v) = \lambda h(u, v)$ and $h(u, \lambda v) = \overline{\lambda} h(u, v)$
4. $h(u, v) = \overline{h(v, u)}$
5. $h(u, u) > 0$ if $u \neq 0$

Decompose h into real and imaginary parts, $h(u, v) = \langle u, v\rangle - i\omega(u, v)$ (where $\langle u, v\rangle$ is the real part and ω is the imaginary part)

Lemma 1.2. *$\langle\, ,\, \rangle$ is a scalar product on V, and ω is a simpletic form, i.e., skew-symmetric.*

Note. $-\langle\, ,\, \rangle$ determines ω and conversely

Proof. By the property 4. $\langle\, ,\, \rangle$ is real symmetric and ω is skew-symmetric. $\langle u, u\rangle = h(u, u) > 0$ so $\langle\, ,\, \rangle$ is scalar product

Let $u_0 \in V$ such that $\omega(u_0, v) = 0 \forall v \in V$. This equates to $h(u_0, v)$ is real for all v. Also $h(u_0, iv)$ is real for all v, but $h(u_0, iv) = -ih(u_0, v) \in \Im$ therefore, $h(u_0, v) = 0$. Hence $u_0 = 0$, as h is non-degenerate. So ω is non-degenerate.

Now we show $-\langle\, ,\, \rangle$ determines ω: $\omega(u, v) = -\Im h(u, v) = \Im(i^2 h(u, v)) = \Im(ih(iu, v)) = \langle iu, v\rangle$, so $\omega(u, v) = \langle iu, v\rangle$. □

Lemma 1.3. *$\omega(u, iu) > 0$ for all $u \neq 0$.*

Proof. Plug in $v = iu$ in the last part of the previous lemma. □

Definition 1.4. We say that a skew-symmetric form on a complex vector space is *positive* if it has the above property (of lemma 1.3)

$$\text{If } h(iu, iv) = h(u, v) \text{ then } \begin{cases} \omega(iu, iv) = \omega(u, v) \\ \langle iu, iv\rangle = \langle u, v\rangle \end{cases} \qquad (*)$$

Exercise. Prove that a 2-form on ω on V satisfy $(*)$ if and only if it is of type $(1, 1)$

Let V be a $\mathbb{C}$-vector space of $\dim_{\mathbb{C}} V = n = 2k$. Let $z_1, \ldots, z_n$ be coordinates on V and $e_1, \ldots, e_n$ be a basis such that $v = \sum_i z_i e_i$ for $v \in V$. Define $dz_j : \sum z_i e_i \mapsto z_j$ and $d\overline{z_j} : \sum z_i e_i \mapsto \overline{z_j}$ for $1 \leq j \leq n$. Then $dz_1, \ldots, dz_n, d\overline{z}_1, \ldots, d\overline{z_n} \in \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C})$. We have $\dim_{\mathbb{R}} \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C}) = 2 \cdot 2n = 4n$. If $\lambda \in \mathbb{C}$ and $\phi \in \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C})$, define $\lambda\phi : v \mapsto \lambda\phi(v)$. So $\operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C})$ can be viewed as a $\mathbb{C}$-vector space, then $\dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C}) = 2n$.

Exercise. $dz_1, \ldots, dz_n, d\overline{z}_1, \ldots, d\overline{z_n}$ is a basis for $\operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C})$ as a $\mathbb{C}$-vector space

Exercise.

1. An element $\phi \in \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C})$ is $\mathbb{C}$-linear if and only if ϕ can be written as $\phi = \sum_{i=1}^n \alpha_i dz_i$ where $\alpha_i \in \mathbb{C}$.
2. ϕ is anti$\mathbb{C}$-linear map if and only if ϕ can be written as $\phi = \sum_{i=1}^n \beta_i d\overline{z_i}$ where $\beta_i \in \mathbb{C}$.

Let I be the set of $\{i_1 < \cdots < i_k\}$. $dz_I = dz_{i_1} \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_k}$ is a k-linear alternating form V to $\mathbb{C}$. $d\overline{z_I} = d\overline{z_{i_1}} \wedge d\overline{z_{i_2}} \wedge \cdots \wedge d\overline{z_{i_k}}$ is a k-linear alternating form V to $\mathbb{C}$

Definition 1.5. A k-form α on V (with values in $\mathbb{C}$), is of type (p, q), with $p+q = k$ if $\alpha = \sum_{|I|=p,|J|=q} \lambda_{I,J} dz_I \wedge d\overline{z_J}$ for $\lambda_{I,J} \in \mathbb{C}$.

Any k-form α is a sum of forms of type (p, q) for $0 \leq p, q \leq k$ and $p + q = k$. Then $\alpha = \sum_{p+q=k} \alpha^{p,q}$

Example. Let V be of dimension 2

$k = 1$ $\quad$ $(1, 0)$-forms are $\mathbb{C}$-linear maps from $V \to \mathbb{C}$

$\quad$ $(0, 1)$-forms are anti-$\mathbb{C}$-linear maps from $V \to \mathbb{C}$

$k = 2$ $\quad$ $(2, 0)$-forms are $dz_1 \wedge dz_2$

$\quad$ $(1, 1)$ forms are spanned by $dz_i \wedge d\overline{z_j}$ for $i, j \in \{1, 2\}$

$\quad$ $(0, 2)$-forms are $d\overline{z_1} \wedge d\overline{z_2}$

Exercise. The type of a form does not depend on the choice of basis.

Example. Let $V = \mathbb{C}^n$, $z_i = x_i + iy_i$ then

$$\begin{aligned} dx_1 \wedge dz_2 &= \frac{dz_1 + d\overline{z_1}}{2} \wedge dz_2 \\ &= \underbrace{\frac{dz_1 \wedge dz_2}{2}}_{(2,0)-\text{form}} + \underbrace{\frac{d\overline{z_1} \wedge dz_2}{2}}_{(1,1)-\text{form}} \end{aligned}$$

Example. If X is a complex surface, z_1, z_2 are local coordinate on X, then a 2-form is a combination

- $dz_1 \wedge dz_2$ a $(2, 0)$-form
- $\begin{cases} dz_1 \wedge d\overline{z_2} \\ dz_1 \wedge d\overline{z_1} \\ dz_2 \wedge d\overline{z_2} \\ dz_2 \wedge d\overline{z_1} \end{cases}$ are $(1, 1)$-forms
- $d\overline{z_1} \wedge d\overline{z_2}$ a $(0, 2)$-form

Summary: If h is a hermitian form, $\omega = -\Im h$ is a $(1, 1)$-form and is positive (i.e, $\omega(u, iu) > 0$). Conversely if a $(1, 1)$ form is positive it arises as $\omega = -\Im h$ for some hermitian form h.

1.2 Hermitian and Kähler metric on Complex Manifolds

Let M be a complex manifold.

Convention: Each tangent space of M, T_xM is a complex vector space and write J (or J_x) for the endomorphism $J_x : T_xM \to T_xM$ defined by $v \mapsto iv$. ($J^2 = -\operatorname{id}$)

Definition 1.6. A *hermitian metric* on M is the following. For each $x \in M$, h_x is a hermitian metric on T_xM and h_x is C^∞ on M.

So as before we can write $h = \langle\, , \rangle - i\omega$. The $\langle\, , \rangle$ is a Riemannian metric on M and ω is a $(1,1)$ form on M

Definition 1.7. We say h is Kähler if ω is closed, i.e., $d\omega = 0$.

Example.

- If $\dim_{\mathbb{C}} M = 1$, that is M is a Riemann surface, then any hermitian metric is Kähler.: Why? $d\omega$ by definition is a 3-form on a 2-dimension $\mathbb{R}$-manifold, so it must be zero.

- $\partial, \overline{\partial}$ operators: If $f : M \to \mathbb{C}$ is a function. df is a 1-form and $df_x : T_xM \to \mathbb{C}$. We can decompose as $df_x = \underbrace{\partial f_x}_{\mathbb{C}-\text{linear}} + \underbrace{\overline{\partial} f_x}_{\mathbb{C}-\text{antilinear}}$. So $df = \partial f + \overline{\partial} f$. ∂ and $\overline{\partial}$ extend to operators from $\Omega^k \to \Omega^{k+1}$ (where Ω^k is the $\mathbb{C}$-value k-forms) defined by

$$\begin{aligned} \partial(\alpha \wedge \beta) &= \partial\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \partial\beta \\ \overline{\partial}(\alpha \wedge \beta) &= \overline{\partial}\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \overline{\partial}\beta \end{aligned}$$

  Exercise. If α is a (p,q)-form (it is of type (p,q) at each point), then $d\alpha$ is the sum of a $(p+1,q)$ form and a $(p,q+1)$ form: ∂ is the $(p+1,q)$ form and $\overline{\partial}$ is the $(p,q+1)$piece

- $M = \mathbb{P}^n_{\mathbb{C}}$: We define a close positive (that is positive on each point on M) $(1,1)$-form (it must be the imaginary part of a hermitian metric) is defined by $\omega_{[z]} = \frac{1}{2\pi i}\partial\overline{\partial}\log(||z||^2)$. Check that it is well defined, (does not define on the affine piece): hint: if $f : U \to \mathbb{C}^*$ is holomorphic, check that $\partial\overline{\partial}\log|f|^2 = 0$.

- If $T = \mathbb{C}^n/\Lambda$, Λ a lattice of $\mathbb{C}^n$, then any constant coefficient metric is Kähler.

- If (M,h) is Kähler, and $\Sigma \subset M$ is a $\mathbb{C}$-submanifold. Then $(\Sigma, h|_\Sigma)$ is Kähler. As $d(\omega|_\Sigma) = d\omega|_\Sigma \Rightarrow \omega|_\Sigma$ is closed.

Lemma 1.8. *Let M be a complex manifold of $\mathbb{C}$-dimension n with hermitian metric h. The Riemannian volume form of $\langle\, , \rangle$ is equal to $\frac{\omega^n}{n!}$.*

(If V is $\mathbb{C}$-vector space with $\mathbb{C}$-basis $e_1, \dots, e_n$ then $e_1, Je_1, e_2, Je_2, \dots, e_n, Je_n$ is a positive real basis. That is it has a canonical basis)

Corollary 1.9. *Let M be a closed complex manifold, i.e., compact with no boundary. Then $\forall k \in \{1, \dots n\}$, $\omega^k = \underbrace{\omega \wedge \cdots \wedge \omega}_{k\text{ times}}$ is closed and non-zero in cohomology, i.e., ω is not exact.*

Proof. If $\omega^k = d\alpha$ for some α, then $\omega^n = \omega^k \wedge \omega^{n-k} = d\alpha \wedge \omega^{n-k} = d(\alpha \wedge \omega^{n-k})$. Hence by Stoke's theorem $\int_M \omega^n = 0$, but $\int_M \frac{\omega^n}{n!} = \operatorname{Vol}(M) > 0$, hence contradiction. ☐

So $H^{2n}_{\mathrm{DR}}(M, \mathbb{R}) \neq 0$

Corollary 1.10. *If M is compact, Kähler and $\Sigma^p \subset M^n$ closed $\mathbb{C}$-submanifold, then the homology class $[\Sigma] \in H_{2p}(M)$, the fundamental class of Σ, is non-zero.*

Proof. $0 < \int_\Sigma \frac{\omega^p}{p!} = \mathrm{Vol}_h \Sigma$, then Σ is not homologeous to 0. □

Exercise. If X is a compact manifold and $\dim_{\mathbb{C}} X \geq 2$ and h a Kähler metric, and $\phi : X \to \mathbb{R}^*_+$. Prove that ϕh is Kähler if and only if ϕ is constant.

1.3 Characterisations of Kähler metrics

Let (M, h) be a complex manifold with hermitian metric. Recall that ∇ is the Levi-Civita connection of $\Re(h) = \langle \, , \, \rangle$ which is a Riemannian metric.

Theorem 1.11. *The following are equivalent:*

1. *h is Kähler*
2. *For any vector field X on $U \subset M$ (open set) then $\nabla(JX) = J(\nabla X)$*

Proof. 2. $\Rightarrow$ 1. By definition of Levi-Civita connection $d\langle X_1, X_2\rangle = \langle \nabla X_1, X_2\rangle + \langle X_1, \nabla X_2\rangle$,

$$\begin{aligned} d\omega(X_1, x_2) &= d\langle JX_1, X_2\rangle \\ &= \langle \nabla JX_1, X_2\rangle + \langle JX_1, X_2\rangle \\ &= \langle J\nabla X_1, X_2\rangle + \langle JX_1, X_2\rangle \end{aligned}$$

so $d\omega(X_1, X_2) = \omega(\nabla X_1, X_2) + \omega(X_1, \nabla X_2)$ $(*)$.

$d\omega(X_0, X_1, X_2) = X_0 \cdot \omega(X_1, X_2) - X_1 \cdot \omega(X_0, X_1) + X_2 \cdot \omega(X_0, X_1) - \omega([X_0, X_1], X_2) + \omega(X_0, [X_1, X_2]) + \omega([X_0, X$

Use $(*)$ and $\nabla_X Y - \nabla_Y X = [X, Y]$ to show that $d\omega(X_1, X_2, X_3) = 0$

1. $\Rightarrow$ 2. Not done

□

1.4 The Hodge decomposition

We want to construct a decomposition of the de Rham cohomology group $H^K_{\mathrm{DR}}(M, \mathbb{C})$ ($\mathbb{C}$-valued differential forms) of a compact Kähler manifold.

If $p + q = k$, we define $H^{p,q}(M) \subset H^k(M)$ by $H^{p,q}(M)$ =subspaces of class $[\alpha]$ such that α can be represented by a closed form of type (p, q), i.e., there exists β of type (p, q) closed such that $\alpha - \beta$ is exact

Our goal:

Theorem 1.12. *If M is compact Kähler, then $H^k(M) = \oplus_{p+q=k} H^{p,q}(M)$. If α is a closed form (on a complex manifold) and if $\alpha = \sum \alpha^{p,q}$ is its decomposition. A priori, the $\alpha^{p,q}$ need not be closed*

Example. $X = (\mathbb{C}^2 \setminus \{0\})/(v \mapsto \frac{1}{2}v)$. Then $H^1(X) \neq 0$ but $H^{1,0}(X)$ and $H^{0,1}(X)$ are zero.

Hodge Theory:

Let $(M, \langle\, ,\, \rangle$ be Riemannian manifold

We need some norms on the space of forms on M, if $e_1, \dots, e_n$ is a orthonormal $\mathbb{R}$-basis of $T_X M$, $e_1^*, \dots, e_n^*$ the dual basis (using $\langle\, ,\, \rangle$ on M) and for each multi-index $\{i_1 < \cdots < i_R\} = I$, let $e_I^* = e_{i_1}^* \wedge \cdots \wedge e_{i_R}^*$. then $\{e_I^*\}_I$ forms a basis of $\Lambda^R (T_x M)^*$ (the space of k forms on $T_x M$).

We declare that $\{e_I^*\}_I$ is orthonormal. This defines a scalar product on $\Lambda^k(T_x M)^*$ (depending only on $\langle\, ,\, \rangle$). We still denote it as $\langle\, ,\, \rangle$. If α, β are k-forms on M we define

$$\langle \alpha, \beta \rangle_{L^2} = \int_M \langle \alpha_x, \beta_x \rangle \operatorname{Vol}$$

Hodge Star Operator

Let $\dim_{\mathbb{R}} M = p$. $\begin{cases} * : \Lambda^k(TM)^* \to \Lambda^{p-k}(TM)^* \\ *^2 = (-1)^{k(p-k)} \end{cases}$. Fix $x \in M$, because $\langle\, ,\, \rangle$ exists on $\Lambda^k(T_x M)^*$ we have the following diagram

$$\begin{array}{ccc} \Lambda^k(T_x M)^* & \overset{\sim}{=\!=\!=} & (\Lambda^k(T_x M)^*)^* \\ & \searrow^{*} & \Big\| \sim \\ & & \Lambda^{p-q}(T_x M^*) \end{array}$$

if β is a $(p-k)$-form and α a k-form with $\alpha \mapsto (\alpha \wedge \beta)/\operatorname{Vol}$ then

$$\langle \alpha, \beta \rangle \operatorname{Vol} = \alpha \wedge *\beta$$

$d : \Lambda^k \to \Lambda^{k+1}$, we want to construct the adjoint d^* of d for $\langle\, ,\, \rangle_{L^2}$. That is we want $d^* : \Lambda^k \to \Lambda^{k-1}$ such that $\alpha \in \Lambda^k$, $\beta \in \Lambda^{k-1}$ then $\langle \alpha, d^*(\beta) \rangle_{L^2} = \langle d\alpha, \beta \rangle_{L^2}$

Claim. If we define d^* on Λ^k by $d^* = (-1)^k *^{-1} d*$ then it works.

Proof. $(\partial\alpha, \beta)_{L^2} = \int_M d\alpha \wedge *\beta$. $d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^k \alpha \wedge d * \beta$, so by Stoke's theorem $0 = \int_M d\alpha \wedge *\beta + (-1)^k \int_M \alpha \wedge d * \beta = \cdots = \langle d\alpha, \beta \rangle_{L^2} - \langle \alpha, d^*\beta \rangle_{L^2}$ □

Definition 1.13. The *Laplacian* $\Delta : \Lambda^k \to \Lambda^k$ is defined by $\Delta = dd^* + d^*d$

Definition 1.14. A k-form α is *harmonic* if $\Delta\alpha = 0$

Lemma 1.15. $\langle \Delta\alpha, \alpha \rangle = |d\alpha|_{L^2}^2 + |d^*\alpha|_{L^2}^2 = \langle d\alpha, d\alpha \rangle_{L^2} + \langle d^*\alpha, d^*\alpha \rangle_{L^2}$ *and* $\langle \Delta\alpha, \beta \rangle = \langle \alpha, \Delta\beta \rangle$

Proof. Exercise (formal) □

Corollary 1.16. $\Delta\alpha = 0$ *if and only if* $d\alpha = 0$ *and* $d^*\alpha = 0$*, i.e., harmonic forms are closed.*

Theorem 1.17. *Any smooth k-form α on M can be written as a sum of a harmonic one plus the Laplacian of another form*

The theorem says that for any α, there exists α_0 harmonic and β a k-form such that $\alpha = \alpha_0 + \Delta\beta$
So we have a map Harmonic k-forms$\to H^k_{\mathrm{DR}}(M)$

Corollary 1.18. *Any de Rham cohomology class can be represented by a unique harmonic form* $H^k(M) \cong \ker(\Delta : \Lambda^k \to \Lambda^k)$

Proof. Let α be a closed k-form. Write $\alpha = \alpha_0 + \Delta\beta$, α_0-harmonic. So $\alpha = \alpha_0 + dd^*\beta + d^*d\beta$ and since α_0 and $dd^*\beta$ are both closed we have $d^*d\beta$ is also closed. $0 = \langle dd^*d\beta, d\beta\rangle_{L^2} = \langle d^*d\beta, d^*d\beta\rangle_{L^2} = ||d^*d\beta|| = 0$, so $d^*d\beta = 0$. Hence $\alpha - \alpha_0 = d(d^*\beta)$ is exact. Hence $[\alpha] = [\alpha_0]$ so $[\alpha]$ is represented by a harmonic form.

We want to show that if α_0 is harmonic and $[\alpha_0] = 0$ then $\alpha_0 = 0$. Let $\alpha_0 = d\gamma$, then $0 = \Delta\alpha_0$ implies $d^*\alpha_0 = 0$. So $d^*d\gamma = 0$, hence $\langle d^*d\gamma, \gamma\rangle_{L^2} = 0 = ||d\gamma||^2_{L^2}$, so $d\gamma = \alpha_0 = 0$ □

We assume now that M is Kähler, $\langle\, ,\,\rangle = \Re(h)$ and h is a Kähler metric.

Theorem 1.19. *In this case the Laplacian preserved the type of forms, that is $\Delta(A^{p,q}) \subset A^{p,q}$ where $A^{p,q}$ is the space of $(p+q)$-forms of type (p,q)*

Corollary 1.20. *The Hodge decomposition exists*

Proof. α is harmonic so $\Delta\alpha = 0$. Write $\alpha = \sum \alpha^{p,q}$ so $\Delta\alpha = \sum \Delta\alpha^{p,q}$. So $\Delta\alpha^{p,q} = 0$, hence $\Delta\alpha^{p,q}$ are harmonic, thence they are closed. So $[\alpha] = \sum[\alpha^{p,q}]$, therefore the $H^{p,q}$ span $H^k(M, \mathbb{C})$

Check that this is a direct sum. □

2 Ricci Curvature and Yau's Theorem

Let $(M, \langle\, ,\,\rangle$ be a Riemannian manifold, ∇ the Levi-Civiti connection

Curvature tensor of M

Let X, Y, Z be vector fields on open set of M.

$$\nabla_Y(\nabla_X Z) - \nabla_X(\nabla_Y Z) - \nabla_{[X,Y]}Z \ (*)$$

Exercise. In Euclidean space, $X, Y, Z : U \to \mathbb{R}^m, \nabla Z = dZ$, then $(*) = 0$

Fact. *$(*)$ is a tensor: The value of $(*)$ at $x \in M$ depends only on $X(x), Y(x), Z(x)$, this means that $(*) = R(X,Y)(Z)$ where $R(X,Y)$ is the endomorphism of T_xM. We call R the* curvature tensor. *It is a bilinear map $T_xM \times T_x \to \mathrm{End}(T_xM)$*

1. *$R(X,Y) = -R(Y,X)$*

2. *$R(X,Y)$ is skew-symmetric for $\langle\, ,\,\rangle$, i.e., $\langle R(X,Y)(Z), T\rangle = -\langle Z, R(X,Y)(T)\rangle$*

   Part 1. tells us we can think of R as 2-form with values in the space of symmetric endomorphism of T_xM. If $p = \dim_{\mathbb{R}} M$ then skew-sym(T_xM) has dimension $\frac{p(p-1)}{2} \times \frac{p(p-1)}{2}$.

   The Ricci tensor of M will be (on each point $x \in M$) a symmetric bilinear from on T_xM. If X, Y are tangent vectors Ricci$(X,Y) := \mathrm{Tr}(R(X,-),(Y))$ (i.e., $\mathrm{Tr}(Z \mapsto R(X,Z)(Y))$

3. *$\langle R(X,Y)(Z), T\rangle = \langle R(Z,T)(X), (Y)\rangle$*

Lemma 2.1. Ricci *is symmetric*

Proof. Let $e_1, \ldots, e_p$ be orthonormal basis of T_xM. Then

$$\begin{aligned} \mathrm{Ricci}(X,Y) &= \sum_i \langle R(X,e_i)(Y), e_i\rangle \\ &= \sum_i \langle R(Y,e_i)(X), e_i\rangle \\ &= \mathrm{Ricci}(Y,X) \end{aligned}$$

□

Next we assume M is Kähler.

Exercise. Prove that $R(JX, JY) = R(X, Y)$ (use the fact that $\nabla JX = J\nabla X$), i.e., that R is of type (1,1)

Let $h = \langle\,,\rangle - i\omega$ be a Hermitian metric. We transform Ricci (a symmetric object) into something skew-symmetric

Definition 2.2. The Ricci form of the Kähler metric is $\gamma_\omega(X, Y) = \mathrm{Ricci}(JX, Y)$

Proposition 2.3. *γ_ω is skew-symmetric and a $(1,1)$-form*

Proof. γ_ω is a $(1,1)$-form because $\gamma_\omega(JX, JY) = \gamma_\omega(X, Y)$

$$\begin{aligned}\gamma_\omega(Y, X) &= \mathrm{Ricci}(JY, X)\\ &= \mathrm{Ricci}(-Y, JX)\\ &= -\gamma_\omega(X, Y)\end{aligned}$$

□

How to relate γ_ω to the 1st Chern Class of M?

We will define the 1st Chern Class of a holomorphic line bundle $L \to M$. $c_1(L) \in H^2(M, \mathbb{R})$ (actually $c_1(L)$ lives in $H^2(M, \mathbb{Z})$, we simply look at its image in $H^2(M, \mathbb{R})$). Let h be a hermitian metric on L. If s is a local holomorphic section without zeroes on some open set U, we define $\Omega = \frac{1}{2\pi i}\partial\overline{\partial}\log h(s, s)$

1. Ω does not depend on s, (i.e., $\partial\overline{\partial}\log h(s_1, s_1) = \partial\overline{\partial}\log h(s_2, s_2)$ is s_1 and s_2 are two non-zero sections on U)
2. Ω is globally defined
3. The cohomology class of Ω does not depend on h (any other hermitian metric on L is of the form $h' = fh$ for $f > 0$, $\Omega' = \Omega + \underbrace{\frac{1}{2\pi i}\partial\overline{\partial}\log f^2}_{\text{is exact}}$)

We define $c_1(L)$ to be the class of Ω. Now if M is a complex manifold its 1st Chern Class is that of the bundle $\Lambda^p TM \to M$ (where $p = \dim_{\mathbb{C}} M$)

Exercise. Let $L \to \mathbb{P}^n_{\mathbb{C}}$ be the tautological line bundle. L can be endowed with the restriction of the metric $\mathbb{C}^{n+1}$. Compute Ω as given above, you should find the negative of the example of Kähler metric of $\mathbb{CP}^n$ given earlier.

On a Kähler manifold $R(X, Y)$ is $\mathbb{C}$-linear, hence skew hermitian.

Proposition 2.4. $\gamma_\omega(X, Y) = -i\,\mathrm{Tr}_{\mathbb{C}}\, R(X, Y)$

Corollary 2.5. *γ_ω is closed* $\left[\frac{\gamma_\omega}{2\pi}\right] = -c_1(M)$

Let ω be a Kähler form on V. Any $(1,1)$-form α on V can be written as $\alpha = \lambda v + \beta$ ($\lambda \in \mathbb{R}$ or $\mathbb{C}$), where β satisfies $\beta \wedge \omega^{n-1} = 0$. (If β satisfies this we say that β is *primitive*)

Corollary 2.6. *Let (M, h) be Kähler. Then $\langle\, ,\, \rangle$ has zero* Ricci *curvature* $\iff$ $R \wedge \omega^n = 0$ *(equivalent to saying R is primitive 2-form).*

If (M, h) is Kähler and if $c_1(M) = 0$, then γ_ω is cohomologeous to zeroes.

Theorem 2.7 (Calabi-Yau). *If (M, ω) is Kähler, $c_1(M) = 0$, then there exists a unique Kähler metric $h_0 = \langle\, ,\, \rangle - i\omega_0$ such that $\begin{cases} [\omega_0] = [\omega] \\ \gamma_{\omega_0} = 0 \end{cases}$. In other words, there is a unique metric with 0 Ricci curvature and cohomologeous to ω.*

3 Hodge Structure

Let M be a finitely generated free module ($M \cong \mathbb{Z}^l$)

Definition 3.1. A *Hodge structure of weight k* on M is a decomposition $M \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_{p+q=k} V^{p,q}$ such that $V^{p,q} = \overline{V^{q,p}}$

Remark.
- $M \otimes \mathbb{C} = M \otimes \mathbb{R} + iM \otimes \mathbb{R}$, so we have an involution $a + ib \mapsto a - ib$ this is the conjugation which appears in the definition.
- In general we assume $V^{p,q} = 0$ if $p < 0$ or $q < 0$

Example. If (M, h) is compact Kähler, $H^k * M, \mathbb{Z})$/Torsion has a weight k Hodge structure. The complexification of $H^k(X, \mathbb{Z})$/Tor is $H^k(X, \mathbb{C})$ and we have the decomposition on $H^k(X, \mathbb{C})$

Definition 3.2. A *polarization* for a Hodge structure of weight k on M is a bilinear form $Q : M \times M \to \mathbb{Z}$ which is

1. Symmetric for k even and skew-symmetric for k odd
2. $Q_{\mathbb{C}} M \otimes \mathbb{C} \times M \otimes \mathbb{C} \to \mathbb{C}$ satisfies $Q_{\mathbb{C}}(\alpha, \overline{\beta}) = 0$ if $\alpha \in V^{p,q}, \beta \in V^{p',q'}$ and $p \neq p'$
3. $\alpha \in V^{p,q} \setminus \{0\}$, $(-1)^{\frac{k(k-1)}{2}}(-1)^q i^k Q(\alpha, \overline{\alpha}) > 0$

Example. $M = H^k(X, \mathbb{Z})/\text{Tor}$, $Q(\alpha, \beta) = \int_X \omega^{n-k} \wedge \alpha \wedge \beta$ (is integral value since $[\omega]$ is integral). This satisfy 1. and 2. but not 3. in general

Proposition 3.3. *A weight 1 Hodge structure is the same thing as a complex torus (a polarised weight 1 Hodge structure is the same thing as an Abelian Variety)*

Proof. $M = \mathbb{Z}^k$, $M \otimes \mathbb{C} = A \oplus \overline{A}$. Consider $v \in M$, then its decomposition must be $(a, \overline{a})$ (since v is real). The projection $\pi : M \otimes \mathbb{C} \to A$ is injective on $\mathbb{Z}^k$. $\pi(\mathbb{Z}^k) \subset A$ (exercise: $\pi(\mathbb{Z}^k)$ is discrete, so its a lattice in A). Then $A/\pi(\mathbb{Z}^k)$ is the complex torus. □

If X is a K3 surface, we will see that $M = H^2(X, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{22}$, $H^2(X, \mathbb{Z}) = H^{2,0} + H^{1,1} + H^{0,2}$ of dimension $1, 20, 1$ respectively.

Lemma 3.4. *If M and the intersection form are given, then the Hodge structure is determined by $H^{2,0}$. In particular, for a K3 surface the Hodge structure is determined by a point in $\mathbb{P}^{21} \subset \mathbb{P}(H^2(X, \mathbb{C}))$. This points lives in the quadric defined by $\int_X \alpha \wedge \alpha = 0$*

Exercise. If β is a $(1,1)$-form then $\beta \wedge \overline{\beta}$ is semi-positive.

Part II
Introduction to Complex Surfaces and K3 Surfaces (Gianluca Pacienza)

References:

Barth,Peters, Vand De Ven: *Compact Complex Surfaces*

Beauville: *Surfaces algebriques Complexes*

Miranda: *An overview of algebraic surfaces (*Free on the internet)

*: *Geometry des surfaces K3*

4 Introduction to Surfaces

We assume X is Kähler for this whole part

4.1 Surfaces

Definition 4.1. A *compact complex surface* (or more simply a surface) X is compact, connected, complex manifold of $\dim_{\mathbb{C}} X = 2$

Example. $F \in \mathbb{C}[x_0, \ldots, x_3]$ homogeneous. $X := \{F = 0\} \subset \mathbb{P}^3$ (of course $F = \frac{\partial F}{\partial x_0} = \cdots = \frac{\partial F}{\partial x_3} = 0$ has no solutions)

More generally if $F_1, \ldots, F_{n-2} \in \mathbb{C}[x_0, \ldots, x_n]$ homogeneous polynomial of degree $d_1, \ldots, d_{n-2}$ such that $\left(\frac{\partial F_i}{\partial x_j}(p)\right)_{i,j}$ has maximal rank at each $p \in X$. (X is called *complete intersection* of multiple degree $(d_1, \ldots, d_{n-2})$)

Note. If $\sum d_i = n + 1$ then X is a K3 surface

Definition 4.2. A surface is called *algebraic* if its field $M(X)$ of meromorphic function satisfies

1. $\forall p \neq q \in X$, $\exists f \in M(X)$ such that $f(p) \neq f(q)$
2. $\forall p \in X$, $\exists f, g \in M(X)$ such that (f, g) gives local coordinates of X at p.

Example.

1. If $X \subset \mathbb{P}^n$ is a surface then it is algebraic, since the ratios $\frac{x_i}{x_j}$ of homogeneous coordinates on $\mathbb{P}^n$ restricted to X satisfies 1 and 2 (of Definition 4.2)
2. $T = \mathbb{C}^2/\Lambda$ a complex torus of $\dim 2$. A “random” choice of Λ will lead to a non-algebraic surface
3. We will see that a “random” K3 surfaces is non-algebraic.

4.2 Forms on Surfaces

Definition 4.3. A *differentiable* 1*-form* (or C^∞) ω on a surface X is locally an expression:

$$f_1(z, w)dz + f_2(z, w)d\overline{z} + g_1(z, w)dw + g_2(z, w)d\overline{w}$$

where (z, w) are local coordinates and f_i, g_i are C^∞ functions (plus patching conditions)

Remark. Since coordinate change preserves $\partial z, \partial \overline{z}, \partial w, \partial \overline{w}$ the type is well defined:

$(1,0)$ type: $f dz + g dw$

$(0,1)$ type: $f d\overline{z} + g d\overline{w}$

Definition 4.4. $(n = 1, 2, 3, 4)$ A C^∞ n-form ω on a surface X is locally a linear combination of expressions of the form $f(z, \overline{z}, w, \overline{w}) d\alpha_1 \wedge \cdots \wedge d\alpha_n$ with $d\alpha_i \in \{dz, d\overline{z}, dw, d\overline{w})$ and $f \in C^\infty$ (with the usual rule $d\alpha_i \wedge d\alpha_i = 0$ and antisymmetric) (plus combability conditions)

A type (p, q) means p-times dz or dw and q-time $d\overline{z}$ or $d\overline{w}$

Definition 4.5. A *holomorphic* (and respectively *meromorphic*) n-form is an n-form of type $(n, 0)$ whose coefficients are holomorphic (respectively meromorphic) functions.

Example. $T = \mathbb{C}^2/\Lambda$. If z_1, z_2 are coordinates on $\mathbb{C}$ then $dz_1, dz_2, d\overline{z_1}, d\overline{z_2}$ descend to the quotient

4.3 Divisors

Definition 4.6. A *divisor* is a finite formal sum $D = \sum_{m_i \in \mathbb{Z}} m_i Y_i$, $Y_i \subset X$ a codimension 1 subvarieties.

i.e., $D \leftrightarrow \left\{\frac{f_i}{g_i}\right\}_{i \in I}$, f_i, g_i local holomorphic function on U_i such that $(f_i/g_i)/(f_j/g_j)$ has no zeroes or poles on $U_i \cap U_j \neq \emptyset$. Hence locally $D =$(zeroes of f_i) $-$ (zeroes of g_i) (all counted with multiplicities)

Divisors form an abelian group $\mathrm{Div}(X)$, $D = \sum_i m_i Y_i, E = \sum_i n_i Y_i$ then $D + E = \sum_i (n_i + m_i) Y_i$, equivalently if $D = \left\{\frac{f_i}{g_i}\right\}$ and $E = \left\{\frac{\alpha_i}{\beta_i}\right\}$then $D + E = \left\{\frac{f_i \alpha_i}{g_i \beta_i}\right\}$.

Definition 4.7. If D is defined globally by zeroes and poles of a meromorphic function $f \in M(X)$ then D is called *principal*

$\mathrm{Prime}(X)$ =Subgroup of principal divisors $\leq \mathrm{Div}(X)$

Definition 4.8. $\mathrm{Pic}(X) := \mathrm{Div}(X)/\mathrm{Prime}(X)$.

Equivalently: We say $D_1, D_2 \in \mathrm{Div}(X)$ are *linearly equivalent* if $\exists f \in M(X)$ such that $D_1 - D_2 = \mathrm{div}(f)$. We use the notation, $D_1 \sim D_2$. So we get a group $\mathrm{Div}(X)/\sim$.

(Which we will avoid calling it $\mathrm{Pic}(X)$, as it is abusive language if X is not algebraic.)

Definition 4.9. If $F : X \to Y$ is a morphism of manifolds and $D = \left\{\frac{f_i}{g_i}\right\} \in \mathrm{Div}(Y)$ then the *pull-back* of D is $F^*D = \left\{\frac{f_i \circ F}{g_i \circ F}\right\}$

The exponential sequence

We have the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0$$

where $\mathcal{O}_X$ is the sheaf of holomorphic functions on X and $\mathcal{O}_X^*$ is the sheaf of non-vanishing holomorphic functions on X.

by taking the long exact sequence in cohomology we get

$$0 \longrightarrow H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

where $H^1(X, \mathcal{O}_X^*)$ represents $\{$line bundles on $X\}/$isom.

Fact. *$H^{p,q}(X) = H^q(X, \Omega^p_X)$, where Ω^p_X is the sheaf of p-forms which are holomorphic.*

Hence $H^1(X, \mathcal{O}_X) = H^{0,1}$. So $0 \to T \to H^1(X, \mathcal{O}^*_X) \to NS(X) \to 0$, where T is the complex torus of dimension $H^{0,1} = H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$ and $NS(X)$ is the image of c_1 map insider $H^2(X, \mathbb{Z})$ called *Neron-Severi* group of X, and its rank (as a $\mathbb{Z}$-module) is called the *Picard number* of X. It is denoted $\rho(X)$.

4.4 The canonical class

Let X be a surface and ω be a meromorphic 2-form on X. Locally $\omega = \frac{f}{g} dz \wedge dw$ where f and g are local holomorphic functions

Definition 4.10. The *canonical divisor* (associated to ω) is $K_X = \left\{\frac{f}{g}\right\}$ = number of zeroes and poles of ω.

Exercise. Check that if ω_1, ω_2 are two meromorphic 2-forms on X then there exists $f \in M(X)$ such that $\omega_1 = f \cdot \omega_2$.

The above exercise implies that the canonical divisors associated to ω_1 and ω_2 are linearly equivalent. Hence K_X defines a unique class in $\mathrm{Div}(X)/\sim$. This class is the *canonical class* of X

Definition 4.11. Given $D \in \mathrm{Div}(X)$, set $H^0(X, \mathcal{O}_X(D)) := \{f \in M(X) : \mathrm{div}(f) \geq -D\}$ = $\mathbb{C}$-vector space of meromorphic functions with poles bounded by D.

Exercise. Show that $H^0(X, \mathcal{O}_X(K_X)) =: H^0(X, K_X) \xrightarrow{\sim} H^{2,0}(X) = H^2(X, \Omega^2 X)$ = $\mathbb{C}$-vector space of holomorphic 2-forms on X

Definition 4.12. The *genus* of a surface X is $p_g(X) = \dim_{\mathbb{C}} H^0(X, K_X)$. More generally the n-th *plurigenus* of X is $\dim_{\mathbb{C}} H^0(X, nK_X)$

Fact. (Important) *The plurigenus are bimeromorphic invariants of X*

Let $\langle f_0, \ldots, f_n \rangle = H^0(X, \mathcal{O}_X(D))$. Let's define $\phi_D : X \dashrightarrow \mathbb{P}^n$ defined by $x \mapsto [f_0(x) : \cdots : f_n(x)]$ (Note: not defined where, either all f_i vanish at x or one of the f_i has a pole at x)

Definition 4.13. Let X be a surface. Suppose $H^0(X, nK_X) \neq 0$ for some $n > 0$. The *Kodaira* dimension of X is $\mathrm{kod}(X) := \max_{m>0} \dim \mathrm{im}(\phi_{mK_X})$ (if possible) otherwise set $\mathrm{kod}(X) = -\infty$. (So $\mathrm{kod}(X) \in \{-\infty, 0, 1, 2\}$

Example. If X is a compact Riemann Sphere, the Riemann-Roch theorem tells us that $\mathrm{kod}(X) =$
$$\begin{cases} -\infty & X \cong \mathbb{P}^1 \\ 0 & p_g(X) = 1 \\ 1 & p_g(X) \geq 2 \end{cases}$$

The Enriques-Kodaira classifications of surfaces consist of "describing" surfaces according to their Kodaira dimension

Example. In each class:

$\mathrm{kod} = -\infty$ Any complete intersection $X_{(d_1, \ldots, d_{n-2})} \subset \mathbb{P}^n$ with $\sum d_i < n+1$

kod $=0$ Any complete intersection $X_{(d_1,\dots,d_{n-2})} \subset \mathbb{P}^n$ with $\sum d_i = n+1$. You can also take a Torus (with $\dim_{\mathbb{C}} T = 2$)

kod $=2$ Any complete intersection $X_{(d_1,\dots,d_{n-2})} \subset \mathbb{P}^n$ such that $\sum d_i > n+1$

kod $=1$ A surface X fibered over a genus ≥ 2 curve $\mathcal{B}$ with fibers isomorphic to curves of genus 1. (e.g. $B \in k[x_0, x_1, x_2]$ homomorphic, $\deg B = 3$, $K = M(B)$, $g(B) \geq 2$, then $X = \{B' = 0\} \subset \mathbb{P}^2_K$)

4.5 Numerical Invariants

Betti Numbers: $b_i := \operatorname{rk} H^i(X, \mathbb{Z}) = \dim_{\mathbb{R}} H^i(X, \mathbb{R}) = \dim_{\mathbb{C}} H^i(X, \mathbb{C}) = i$th Betti number of X

Euler Number $e := \sum (-1)^i b_i$

Hodge numbers $h^{p,q} := \dim_{\mathbb{C}} H^{p,q} = \dim_{\mathbb{C}} H^q(X, \Omega^q_X)$

They satisfy $h^{p,q} = h^{q,p} = h^{2-p,2-q} = h^{2-q,2-p}$ and $b_k = \sum_{p+q=k} h^{p,q}$ (Hodge decomposition). This gives the Hodge diamond

$$\begin{matrix} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{matrix} \quad \Rightarrow \quad \begin{matrix} & & 1 & & \\ & q & & q & \\ p_g & & h^{1,1} & & p_g \\ & q & & q & \\ & & 1 & & \end{matrix}$$

where q is called the *irregularity* of X. Note that $e = 2 + 2p_g + h^{1,1} - 4g$

4.6 Intersection Number

Let $C_1, C_2 \subset X$ be two irreducible curves. We want to define $C_1 \cdot C_2$. If $C_1 \neq C_2$, set $C_1 \cdot C_2 = \sum_{p \in C_1 \cap C_2} (C_1 \cdot C_2)_p$ where $(C_1 \cdot C_2)_p = \dim \mathcal{O}_{X,p}/(f_1, f_2)$ with $C_i = \{f_i = 0\}$ (locally)

Exercise. Check that $(C_1 \cdot C_2)_p = 1$ if C_1 and C_2 are smooth at p are intersect transversely, i.e. $C_1 \cdot C_2 =$ #points in $C_1 \cap C_2$ counted with the right multiplicities (as usual)

If $C_1 = C_2 = C$. If C is smooth, $C^2 := \deg(N_{C/X})$, the normal bundle of C in X. For the general definition look at references.

4.7 Classical (and useful) results

Thom-Hirzebruch index theorem The index (number of positive eigenvalues minus number of negative eigenvalues) of the intersection product on $H^2(X)$ is equal to $\frac{K_X^2 - 2e}{3}$

Hodge index theorem The intersection product on $H^{1,1} \cap H^2(X)$ has signature $(1, h^{1,1} - 1)$

Noether's formula $12\chi(\mathcal{O}_X) = K_X^2 + e$ where $\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X)$

Riemann Roch Let $D \in \operatorname{Div}(X)$, $\chi(\mathcal{O}_X(D)) = \frac{D \cdot (D - K_X)}{2} + \chi(\mathcal{O}_X)$ where $\chi(L) = h^0(L) - h^1(L) + h^2(L)$ and $L = \mathcal{O}_X(D)$ in our case

Genus formula Let $C \subset X$ be an irreducible curve, then $2p_a(C)-2=(C+K_X)\cdot C$ where $p_a(C)=h^1(\mathcal{O}_C)$ is the arithmetic genus (and is equal to the topological genus of C if C is smooth)

Freedman X_1, C_2 simply connected surface. We have$X_1 \cong X_2$ (homorphically) if and only if $H^2(X_1,\mathbb{Z}) \cong H^2(X_2,\mathbb{Z})$ (isometrically)

5 Introduction to K3 surfaces

Definition 5.1. A surface X is a K3 surface if $K_X=0$ and $b_1(X)=0$

Theorem 5.2. *A K3 surface is always Kähler*

Remark. Since $b_1(X)=2q$ an equivalent definition is $K_X=0$ and $h^1(\mathcal{O}_X)=0$

Noether formula reads $12\cdot(2-0)=0+e$, that is $24=e=2+2+h^{1,1}-0$, hence $h^{1,1}=20$

Exercise. Let X be a K3 surface. Prove that $T_X \cong \Omega^1_X$

A consequence of exercise is that $\dim_{\mathbb{C}} H^1(X,T_X)=20$
We have that the Hodge diamond of a K3 surface is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Fact. *$H_1(K3,\mathbb{Z})$ has no torsion*

Corollary 5.3. *Let X be a K3 surface. $H_1(X,\mathbb{Z})=0$ and $H_2(X,\mathbb{Z})$ is a torsion free $\mathbb{Z}$-module of rank 22*

Proof. $H_1(X,\mathbb{Z})\otimes\mathbb{R}=0$ (since $b_1=0$) so $H_1(X,\mathbb{Z})=0$. By general properties of algebraic topology, we have that the torsion of $H_2(X,\mathbb{Z})$ is isomorphic to the torsion of $H_1(X,\mathbb{Z})$. Hence no torsion. Since $b_2(X)=22$, then $H_2(X,\mathbb{Z})$ is a torsion free $\mathbb{Z}$-module of rank 22 □

A closer look to $H^2(X,\mathbb{Z})$

- $H^2(X,\mathbb{Z})$ is endowed with the intersection form, which is even by the genus formula ($2p_a(C)-2=(C+0)\cdot C=C^2$)
- The intersection form is indefinite (since, by Thon-Hizebucj, the index is -16)
- The intersection form is unimodular (its determinant is ±1) by Poincaré duality

Now we have the following:

Fact. *An indefinite, unimodular lattice is uniquely determined (up to isometry) by its rank, index and parity (i.e., even or not)*

Conclusion: $H^2(\mathrm{K3},\mathbb{Z})=H^{\oplus 3}\oplus(-E_8)^{\oplus 2}$ where

- H is a rank 2 $\mathbb{Z}$-module with form given $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (Hyperbolic plane)
- $E_8 = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_8$ a rank 8 $\mathbb{Z}$-module with the following Dykin diagram

$$\begin{array}{ccccccccccccc} e_1 & & e_2 & & e_3 & & e_4 & & e_5 & & e_6 & & e_7 \\ \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet \\ & & & & & & & & | & & & & \\ & & & & & & & & e_1 & & & & \\ & & & & & & & & \bullet & & & & \end{array}$$

and $(e_i, e_j) = \begin{cases} 2 & i = j \\ -1 & d(e_i, e_j) = 1 \ \ (d(e_i, e_j) \text{ is given by the diagram}) \\ 0 & \text{else} \end{cases}$

(Check that $H^{\oplus 3} \oplus (-E_8)^{\oplus 2}$ has the same rank, index and parity as $H^2(\mathrm{K3}, \mathbb{Z})$)

Note. The sign on H^2 is $(3, 19)$ while the sign on $H^{1,1} \cap H^2$ is $(1, 19)$

We conclude with 3 classes of examples of K3

1. Complete intersections in $\mathbb{P}^n$. Take $X = X_{(d_1,\ldots,d_{n-2})}\mathbb{P}^n$ a complete intersection with surface of multidegree $(d_1, \ldots, d_{n-2})$ such that $\sum d_i = n + 1$. By applying $(n-2)$ times the adjunction formula, we find $K_X = 0$. By applying $(n-2)$ time the Lefschetz hyperplane theorem, we see $H^1(X, \mathbb{Z}) \xrightarrow{\sim} H^1(\mathbb{P}^n, \mathbb{Z}) = 0$. So X is a K3 surface (for example $X_4 \subset \mathbb{P}^3, X_{2,3} \subset \mathbb{P}^4, X_{2,2,2} \subset \mathbb{P}^5, \ldots$)

   Parameter counts: (for $X_4 \subset \mathbb{P}^3$) We have 35 parameters (the complex dimension of the space of degree 4 homogeneous polynomials in 4 variable) minus 16 parameters (the complex dimension of 4×4 invertible matrices). Hence a total of 19 parameters.

2. Double Planes: Take $C = C_6 \subset \mathbb{P}^2$ a smooth sextic plane curve. Let$X$ be the double cover of $\mathbb{P}^2$ branched along C, $X \xrightarrow{\pi} \mathbb{P}^2$ (c.f. [BPHVdV]).

   Theorem 5.4. $K_X = \pi^*(K_{\mathbb{P}^2})$+Ramification $= \pi^*(-3H + \frac{1}{2}C) \sim \pi^*(-3H + \frac{6}{2}H) = 0$

   One also computes that $b_1(X) = 0$, so X is a K3 surface

   Parameter count: 28 parameters (the complex dimension of the space of homogeneous degree 6 polynomial in 3 variable) minus 9 parameters (the complex dimension of invertible 3×3 matrices acting on $\mathbb{P}^2$) then 19 parameters.

3. Kummer Surfaces:

   Let A be a complex torus of $\dim_{\mathbb{C}} 2$. We have an involution $\iota : A \to A$, $a \mapsto -a$. Consider A/ι (i.e identify each point of A with its opposite)

   Bad News: A/ι has 16 singulars points (corresponding to the 16 fixed points of ι) which are exactly the 16 points of order 2 on A

   Good News: We can get rid of them by Blowing up. Let $\epsilon : \widetilde{A} \to A/\iota$ be the blow up at these 16 points.

$$\begin{array}{ccc} \epsilon : \widetilde{A} & \longrightarrow & A \\ \downarrow & & \downarrow \\ \widetilde{A}/\widetilde{\iota} = A' & \to & A/\iota \end{array}$$

Notice: That locally around an order 2 point $\iota : (\alpha, \beta) \mapsto (-\alpha, -\beta)$, the invariants under ι are $\alpha^2, \beta^2, \alpha\beta$. So $A/\iota = \operatorname{Spec} \mathbb{C}[\alpha^2, \beta^2, \alpha\beta] \cong \operatorname{Spec} \mathbb{C}[u, v, w]/(uv - w^2)$. This shows that the singular points of A/ι are ordinary double points. If $\widetilde{\iota} : \widetilde{A} \to \widetilde{A}$ is the extension of ι to $\widetilde{A}$, then one sees that around the exceptional curves $\widetilde{i} : (x, y) \mapsto (x, -y)$. The upshot is that the quotient $X := \widetilde{A}/\widetilde{\iota}$ is smooth.

X is a K3 surface: The 2-form $d\alpha \wedge d\beta$ descend to the quotient, and then lifts to $A' \backslash \{$exceptional curves$\}$. One can check that it extend smoothly to A' without zeroes. As why it has no irregularity $(h^{1,0} = h^0(\Omega^1_X))$, there does not exists a 1 form on A which is invariant under the involution ι.