An introduction to the Hasse principle

through examples

Álvaro González Hernández

Outline of the talk

1. Motivation
2. p-adic numbers
3. The Hasse principle
4. Elliptic curves
5. Advanced topics

$$3x^3 + 4y^3 = 5$$

$$2y^2 = 1 - 17x^4$$

Hasse ?

Let's study some equations

$$3x + 5y = 1 \qquad x, y \in \mathbb{Q}$$

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$$(x, y) = \left(t, \tfrac{1-3t}{5}\right) \quad t \in \mathbb{Q}$$

Let's study some equations

$$x^2 + y^2 = 1 \qquad\qquad x, y \in \mathbb{Q}$$

Setting $x = t$ does not work because $y = \sqrt{1 - t^2}$ may not be rational.

Let's study some equations

$$x^2 + y^2 = 1 \qquad x, y \in \mathbb{Q}$$

By using geometric arguments, we can prove that the solutions are either $(x, y) = (-1, 0)$ or $(x, y) = (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$ with $t \in \mathbb{Q}$

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$$x = 0, \quad y = 0$$

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$$x^2 - 3y^2 = 2 \qquad x, y \in \mathbb{Q}$$

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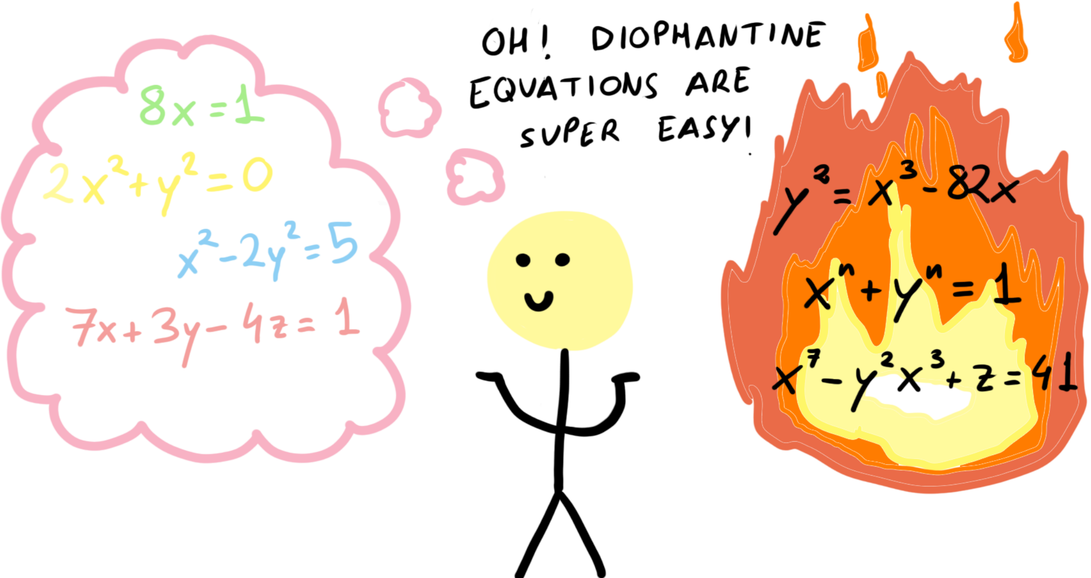
No solutions!

Consider the equivalent equation $X^2 - 3Y^2 = 2Z^2 \qquad X, Y, Z \in \mathbb{Z} \setminus \{0\}$

$X^2 \equiv 2Z^2 \pmod 3$ can only happen if $X \equiv Z \equiv 0 \pmod 3$ because 2 is not a square modulo 3.

This implies that $3|X$, $3|Z$ and $3|Y$. Contradiction!

Let's study some equations



Conclusions of our little study

1. Studying the solutions of Diophantine equations is (generally) **hard**.
2. Finding the **real solutions** of a Diophantine equation can help us determine the existence (or not) of solutions.
3. Considering the homogeneous equations modulo n for appropriate $n \in \mathbb{N}$, we can sometimes determine if an equation does not have integer solutions.

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HOW?

p-adic absolute value

Fix a prime p. Let $x = p^n \frac{a}{b} \in \mathbb{Q}$ non-zero, with $a, b, n \in \mathbb{Z}$ and a, b coprime to p. Then, the ***p*-adic absolute value** of x is defined to be

$$|x|_p = p^{-n}$$

Examples

$$\left| 10 \right|_5 = \frac{1}{5} \qquad \left| \frac{3}{4} \right|_5 = 1 \qquad \left| \frac{1}{125} \right|_5 = 125$$

p-adic numbers

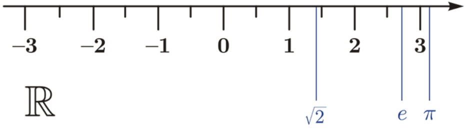
p-adic numbers

The field of ***p*-adic numbers** $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the absolute value $|\ \ |_p$.

- The completions of $\mathbb{Q}$ with respect to a valuation are known as **local fields** of the ring $\mathbb{Q}$. These are either $\mathbb{R}$ or $\mathbb{Q}_p$ for some p prime.
- The field $\mathbb{Q}$ is known as the **global field** of $\mathbb{R}$ and $\mathbb{Q}_p$.

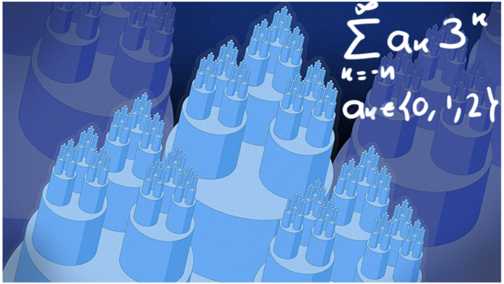
What do local fields look like?

Real numbers



→ Elegant
→ Easy to think of
→ Used by engineers

3-adic numbers

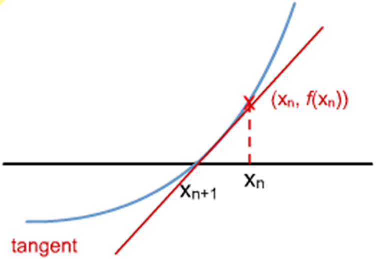


→ Alien looking
→ Mostly used in algebra and number theory

Tools to solving equations

Real numbers

Newton's method



p-adic numbers

Hensel's lemma

polynomial

To solve $x \in \mathbb{Q}_p$ with $f(x) = 0$ we consider the limit of x_n where $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ for appropriate inicial x_0.

Application of Hensel's lemma

Let $b \in \mathbb{Z}$, $x^2 = b$ has a solution $x \in \mathbb{R}$ iff $b \geq 0$. Let's see what happens in $\mathbb{Q}_p$.

Theorem

The equation $x^2 = b$ has a solution $x \in \mathbb{Q}_2$ if and only if b is of the form $b = 2^{2n} b_0$ with $n \in \mathbb{Z}_{\geq 0}$ and $b_0 \equiv 1 \pmod 8$.

For any odd prime p, the equation $x^2 = b$ has a solution $x \in \mathbb{Q}_p$ if and only if b is of the form $b = p^{2n} b_0$ with $n \in \mathbb{Z}_{\geq 0}$ and b_0 a quadratic residue modulo p (this means $b_0 \equiv a_0^2 \pmod p$ for some a_0).

Example of application of this theorem

$x^2 = 17$ for some $x \in \mathbb{Q}_2$, as $17 \equiv 1 \pmod 8$.
$x^2 = 2$ for some $x \in \mathbb{Q}_{17}$, as $6^2 \equiv 2 \pmod{17}$.

Hasse Principle

No solution in $\mathbb{Q}_p$ for some p or no solution in $\mathbb{R}$ $\Rightarrow$ No solution in $\mathbb{Q}$.

Hasse Principle (The converse statement)

If a system of polynomial equations with rational coefficients has a solution in $\mathbb{R}$ and in $\mathbb{Q}_p$ for every prime p, then it has a solution in $\mathbb{Q}$.

Does this principle always hold?

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No

Counterexample to the Hasse principle

Consider the equation $(x^2-2)(x^2-17)(x^2-34)=0$. It has

- Six real solutions $\{\pm\sqrt{2}, \pm\sqrt{17}, \pm\sqrt{34}\}$.
- At least two 2-adic solutions (as $x^2-17=0$ for some $x \in \mathbb{Q}_2$).
- At least two 17-adic solutions (as $x^2-2=0$ for some $x \in \mathbb{Q}_{17}$).
- At least two p-adic solutions for any other prime p (because if 2 and 17 are not quadratic residues modulo p, then 34 is a quadratic residue and so, at least one of the equations

$$x^2-2=0 \qquad x^2-17=0 \qquad x^2-34=0$$

  must have solutions in $\mathbb{Q}_p$).
- No rational solutions.

Hasse principle for curves

Let $\mathcal{C}$ be a curve (smooth, projective, irreducible variety) over $\mathbb{Q}$.

Theorem

Let $\mathcal{C}$ be a curve over $\mathbb{Q}$ of genus 0. Then, it is $\mathbb{Q}$-birationally equivalent either to a line or to a conic section of the form $aX^2 + bY^2 + cZ^2 = 0$, with $a, b, c \in \mathbb{Q}$.

Theorem (Hasse-Minkowski)

Every curve of genus 0 satisfies the Hasse principle.

Hasse principle for curves

For curves with genus $g > 0$, the Hasse principle does not generally apply.

Challenge

Finding counterexamples to the Hasse principle in curves.

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Lind and Reichart (1940)

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Lind and Reichart (1940)

$$2y^2 = 1 - 17x^4$$

$$\updownarrow$$

$$y^2 = x^3 + 17x$$

Selmer (1951)

$$3x^3 + 4y^3 = 5$$

$$\updownarrow$$

$$y^2 = x^3 - 2^8 3^5 5^2$$

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Elliptic curve

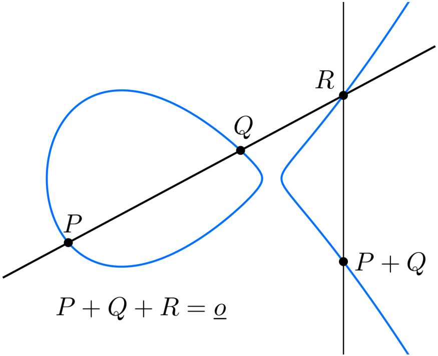
An **elliptic curve** $\mathcal{E}$ is a non-singular curve of genus 1 with a rational point. We denote by $\mathcal{E}(\mathbb{Q})$ its set of rational points.

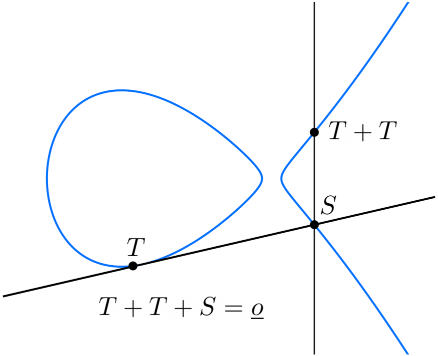
An elliptic curve can always be transformed into a curve that in affine coordinates has equation

$$y^2 = x^3 + Ax + B \qquad \text{with } A, B \in \mathbb{Q}$$

Group Law

We can "sum" points in an elliptic curve and $\mathcal{E}(\mathbb{Q})$ is a group with respect to this operation





Some relevant concepts in elliptic curves

Multiplication-by-m isogeny

$$\begin{aligned}[m]\colon\ \mathcal{E}(\mathbb{Q}) &\longrightarrow \mathcal{E}(\mathbb{Q})\\ P &\longmapsto \underbrace{P+\cdots+P}_{m\text{ terms}}.\end{aligned}$$

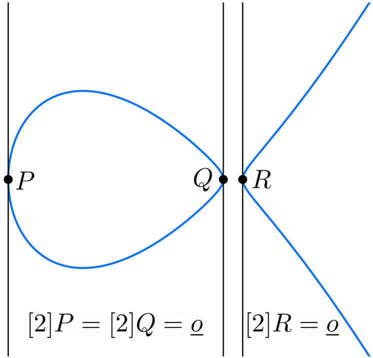
m-torsion subgroup

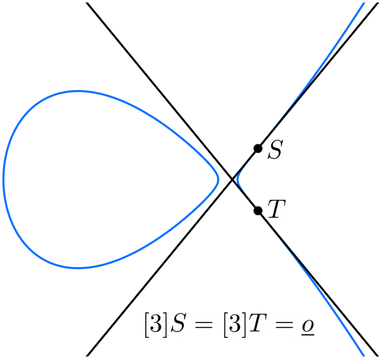
$$\mathcal{E}(\mathbb{Q})[m] = \{P \in \mathcal{E}(\mathbb{Q}) : [m]P = \underline{o}\}.$$

We will denote by $\mathcal{E}_{\text{tors}}(\mathbb{Q})$ the points of finite order in $\mathcal{E}$, i.e.,

$$\mathcal{E}_{\text{tors}}(\mathbb{Q}) = \bigcup_{m=2}^{\infty} \mathcal{E}(\mathbb{Q})[m].$$

Example of torsion points





$\mathcal{E}(\mathbb{Q})$ as a group

Theorem (Mordell-Weil)

Let $\mathcal{E}$ be an elliptic curve. Then, the group $\mathcal{E}(\mathbb{Q})$ is finitely generated and so

$$\mathcal{E}(\mathbb{Q}) \cong \mathcal{E}_{\text{tors}}(\mathbb{Q}) \times \mathbb{Z}^r$$

where $\mathcal{E}_{\text{tors}}(\mathbb{Q})$ is finite, and r is called the **rank** of $\mathcal{E}(\mathbb{Q})$.

Computing the rank of an elliptic curve is **generally hard** and there are many questions that we don´t know about it. For instance,

Unsolved problem

Is the rank of elliptic curves bounded?

Galois cohomology of an elliptic curve

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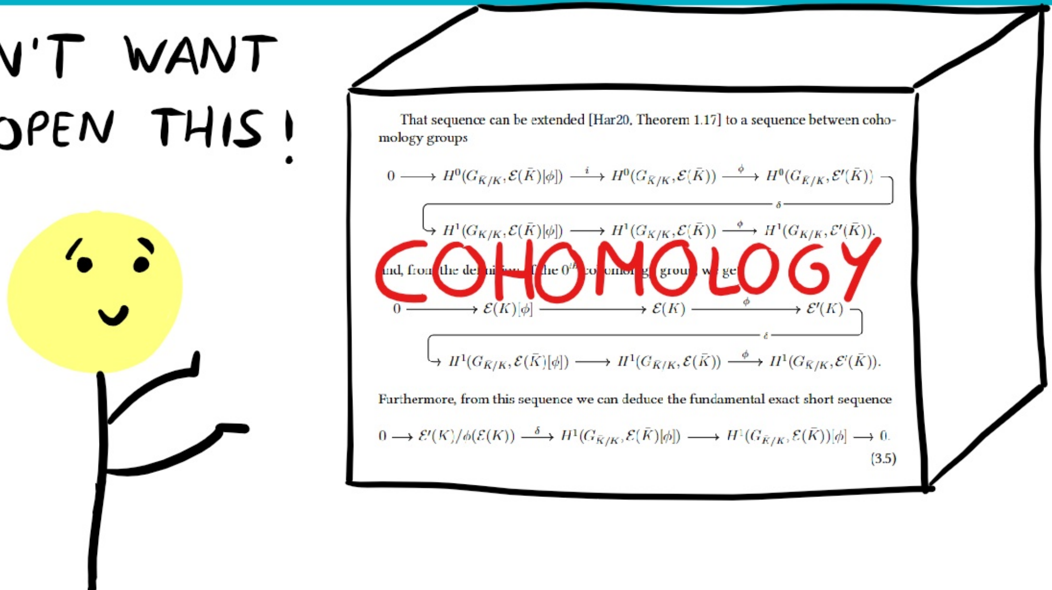


COHOMOLOGY

That sequence can be extended [Har20, Theorem 1.17] to a sequence between cohomology groups

$$0 \longrightarrow H^0(G_{\bar{K}/K}, \mathcal{E}(\bar{K})[\phi]) \xrightarrow{i} H^0(G_{\bar{K}/K}, \mathcal{E}(\bar{K})) \xrightarrow{\phi} H^0(G_{\bar{K}/K}, \mathcal{E}'(\bar{K}))$$
$$\xrightarrow{\delta} H^1(G_{\bar{K}/K}, \mathcal{E}(\bar{K})[\phi]) \longrightarrow H^1(G_{\bar{K}/K}, \mathcal{E}(\bar{K})) \xrightarrow{\phi} H^1(G_{\bar{K}/K}, \mathcal{E}'(\bar{K})).$$

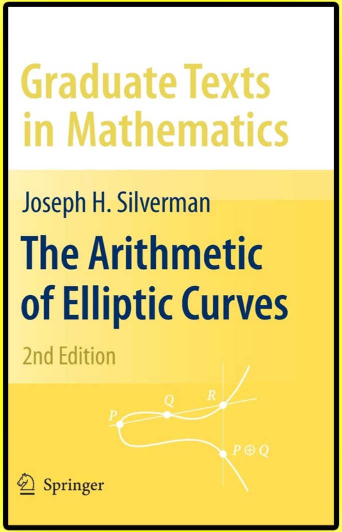
and, from the definition of the 0^{th} cohomology group, we get

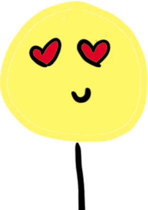
$$0 \longrightarrow \mathcal{E}(K)[\phi] \longrightarrow \mathcal{E}(K) \xrightarrow{\phi} \mathcal{E}'(K)$$
$$\xrightarrow{\delta} H^1(G_{\bar{K}/K}, \mathcal{E}(\bar{K})[\phi]) \longrightarrow H^1(G_{\bar{K}/K}, \mathcal{E}(\bar{K})) \xrightarrow{\phi} H^1(G_{\bar{K}/K}, \mathcal{E}'(\bar{K})).$$

Furthermore, from this sequence we can deduce the fundamental exact short sequence

$$0 \longrightarrow \mathcal{E}'(K)/\phi(\mathcal{E}(K)) \xrightarrow{\delta} H^1(G_{\bar{K}/K}, \mathcal{E}(\bar{K})[\phi]) \longrightarrow H^1(G_{\bar{K}/K}, \mathcal{E}(\bar{K}))[\phi] \longrightarrow 0. \tag{3.5}$$

Galois cohomology





Computing the rank

To find the rank of an elliptic curve, we can study the group $\mathcal{E}(\mathbb{Q})/2\,\mathcal{E}(\mathbb{Q})$, as

$$\mathcal{E}(\mathbb{Q})/2\,\mathcal{E}(\mathbb{Q}) \cong \mathcal{E}(\mathbb{Q})[2] \times (\mathbb{Z}/2\mathbb{Z})^r$$

By the theory of Galois cohomology we get a short exact sequence

$$0 \longrightarrow \mathcal{E}(\mathbb{Q})/2\,\mathcal{E}(\mathbb{Q}) \longrightarrow H^1(\mathcal{E}(\mathbb{Q})[2]) \xrightarrow{\psi} WC(\mathcal{E}(\mathbb{Q}))[2] \longrightarrow 0$$

- $H^1(\mathcal{E}(\mathbb{Q})[2])$ is a finite subgroup of $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2 \times \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$.
- $WC(\mathcal{E}/\mathbb{Q})[2]$ is a group whose elements are **homogeneous spaces.**
- $\ker(\psi) \cong \mathcal{E}(\mathbb{Q})/2\,\mathcal{E}(\mathbb{Q})$ are the elements of $H^1(\mathcal{E}(\mathbb{Q})[2])$ whose images homogeneous spaces **do not have rational points.**

What does this have to do with the Hasse principle?

We can study "**Galois cohomology with respect to local fields**" to get

$$0 \longrightarrow \mathcal{E}(\mathbb{Q})/2\,\mathcal{E}(\mathbb{Q}) \longrightarrow \mathrm{Sel}^{(2)}(\mathcal{E}/\mathbb{Q}) \longrightarrow \text{Ш}(\mathcal{E}/\mathbb{Q})[2] \longrightarrow 0$$

where

- $\mathrm{Sel}^{(2)}(\mathcal{E}/\mathbb{Q})$ is the **2-Selmer group**, which is the subgroup of all elements of $H^1(\mathcal{E}(\mathbb{Q})[2])$ whose images homogeneous spaces **have points in** $\mathbb{R}$ **and in** $\mathbb{Q}_p$ **for every** p.
- $\text{Ш}(\mathcal{E}/\mathbb{Q})$ is the **Tate-Shaferevich** group, which is a group that measures to what extent all possible homogeneous spaces associated to $\mathcal{E}$ satisfy the Hasse principle.

2-Selmer and Tate-Shaferevich groups

$$0 \longrightarrow \mathcal{E}(\mathbb{Q})/2\,\mathcal{E}(\mathbb{Q}) \longrightarrow \mathrm{Sel}^{(2)}(\mathcal{E}/\mathbb{Q}) \longrightarrow Ш(\mathcal{E}/\mathbb{Q})[2] \longrightarrow 0$$

If we are able to prove that $Ш(\mathcal{E}/\mathbb{Q})$ is non-trivial (for instance, by proving that $Ш(\mathcal{E}/\mathbb{Q})[2]$ is non-trivial), then some of the homogeneous spaces of $\mathcal{E}$ are counterexamples of the Hasse principle.

In fact, $2y^2 = 1 - 17x^4$ is a counterexample of the Hasse principle, because it is a homogeneous space associated to the curve $\mathcal{E} : y^2 = x^3 + 17x$, which has $Ш(\mathcal{E}/\mathbb{Q})[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$.

The Tate-Shaferevich group is mysterious

Conjecture

The Tate-Shaferevich group is finite.

If this conjecture is true, then the order of $Ш(\mathcal{E}/\mathbb{Q})$ is a square and so is the order of any of the p-primary components of $Ш(\mathcal{E}/\mathbb{Q})$.

From

$$0 \longrightarrow \mathcal{E}(\mathbb{Q})/2\,\mathcal{E}(\mathbb{Q}) \longrightarrow \mathrm{Sel}^{(2)}(\mathcal{E}/\mathbb{Q}) \longrightarrow Ш(\mathcal{E}/\mathbb{Q})[2] \longrightarrow 0$$

we deduce that

$$\begin{aligned} \mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}}\, Ш(\mathcal{E}/\mathbb{Q})[2] &= \mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}}\, \mathrm{Sel}^{(2)}(\mathcal{E}/\mathbb{Q}) - \mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}}\, \mathcal{E}(\mathbb{Q})/2\,\mathcal{E}(\mathbb{Q}) \\ &= \mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}}\, \mathrm{Sel}^{(2)}(\mathcal{E}/\mathbb{Q}) - \mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}}\, \mathcal{E}(\mathbb{Q})[2] - \mathrm{rank}_{\mathbb{Z}}\, \mathcal{E}(\mathbb{Q}) \end{aligned}$$

Why are we interested in the Hasse principle?

Conjecture (Birch and Swinnerton-Dyer). *The Taylor expansion of $L(C,s)$ at $s=1$ has the form* 1M \$

$$L(C,s) = c(s-1)^r + \textit{higher order terms}$$

with $c \neq 0$ and $r = \operatorname{rank}(C(\mathbb{Q}))$.

In particular this conjecture asserts that $L(C,1)=0 \Leftrightarrow C(\mathbb{Q})$ is infinite.

Remarks. 1. There is a refined version of this conjecture. In this version one has to define Euler factors at primes $p|2\Delta$ to obtain the completed L-series, $L^*(C,s)$. The conjecture then predicts that $L^*(C,s) \sim c^*(s-1)^r$ with

$$c^* = |Ш_C| R_\infty w_\infty \prod_{p|2\Delta} w_p / |C(\mathbb{Q})^{\text{tors}}|^2.$$

Here $|Ш_C|$ is the order of the Tate–Shafarevich group of the elliptic curve C.

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Here it is!

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Thank you!