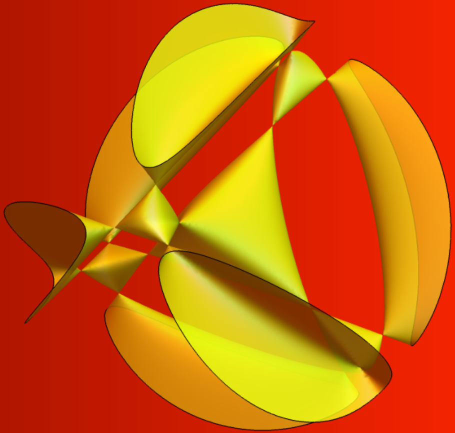
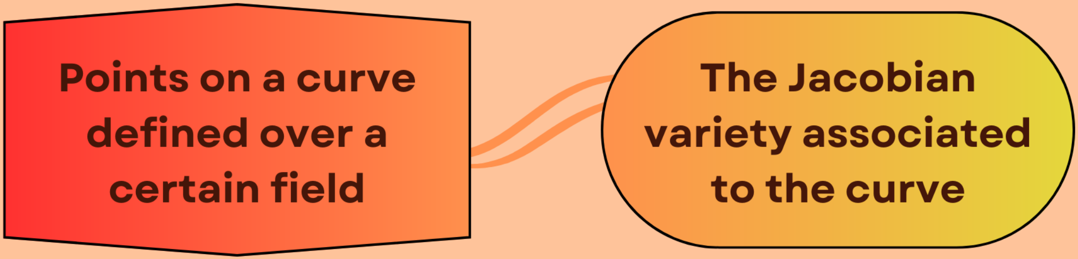
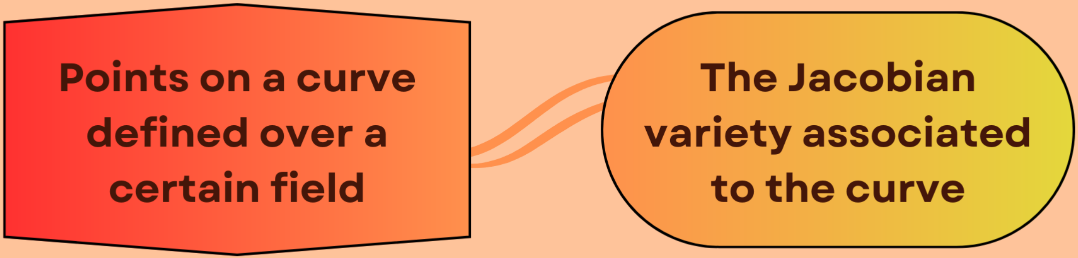
ALVARO GONZALEZ HERNANDEZ
University of Warwick



Explicit models of Kummer surfaces in characteristic two

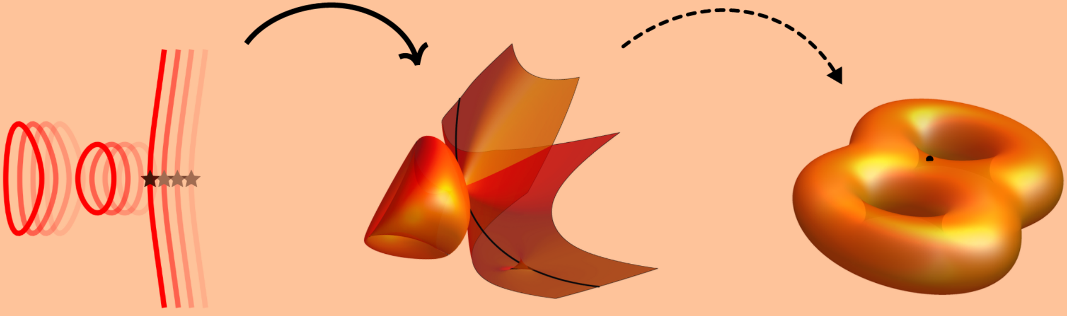

Points on a curve defined over a certain field
The Jacobian variety associated to the curve



Given a curve, can we compute an explicit model of its Jacobian as a projective variety?

In theory, yes!

In theory, yes!

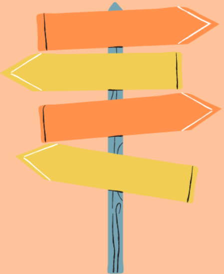


$$\mathcal{C}^{(g)} = \underbrace{\mathcal{C} \times \cdots \times \mathcal{C}}_{g} \; / \; S_g$$

Jacobian variety

In practice...

Models that work for all curves, but have complicated equations



Models that only work for special classes of curves, but have simpler equations

In this talk

Models that work for all curves, but have complicated equations

Models that only work for special classes of curves, but have simpler equations

How to compute a explicit model of the Jacobian

Let $\mathcal{C} : y^2 + h(x)y = f(x)$ be a hyperelliptic curve of genus $g \geq 1$ where $f(x), h(x) \in k[x]$, $\deg f(x) = 2g+2$ and $\deg h(x) \leq g+1$.

The curve has two different points at infinity that I will denote by ∞_+ and ∞_-. Then,

$$\Theta_+ = \underbrace{C \times \cdots \times C}_{g-1} \times \{\infty_+\} \quad \text{and} \quad \Theta_- = \underbrace{C \times \cdots \times C}_{g-1} \times \{\infty_-\}$$

define divisors of $\mathcal{C}^{(g)}$ and an embedding of the Jacobian into projective space is given by $\mathcal{L}(2(\Theta_+ + \Theta_-))$.

But there is a drawback...

The embedding by $\mathcal{L}(2(\Theta_+ + \Theta_-))$ is given by the intersection of **many** conics:

Genus	1	2	3	$\cdots$	g
$\mathbb{P}^n$ in which it embeds	3	15	63	$\cdots$	$4^g - 1$
Number of conics	2	72	1568	$\cdots$	$2^{2g-1}(2^g - 1)^2$

Kummer varieties

Kummer variety

Let $\mathcal{A}$ be an Abelian variety and let ι be the involution in $\mathcal{A}$ that sends an element to its inverse. Then, the **Kummer variety** associated to $\mathcal{A}$, $\mathrm{Kum}(\mathcal{A})$ is the quotient variety $\mathcal{A}/\iota$.

Fact

For $g > 1$, $\mathcal{A}[2]$ is the set of all fixed points under the action of ι and these points are singular points of $\mathrm{Kum}(\mathcal{A})$.

Examples of Kummer varieties

Suppose that k is a field of characteristic different than 2.

- If the dimension of $\mathcal{A}$ is 2, $\mathrm{Kum}(\mathcal{A})$ is a surface described by a quartic in $\mathbb{P}^3$ with 16 (A_1) nodal singularities.
- Generally, if the dimension of $\mathcal{A}$ is g, $\mathrm{Kum}(\mathcal{A})$ can be found as an intersection in $\mathbb{P}^{2^g-1}$.

Why are Kummer varieties relevant?

- Their models are considerably easier.
- They are **not** Abelian varieties, so they do not have a group law. However, they inherit a *pseudo-group law*.
- For a hyperelliptic curve $\mathcal{C}$, the projective embedding of the Kummer variety associated to the Jacobian of $\mathcal{C}$ is given by $\mathcal{L}(\Theta_+ + \Theta_-)$.

1. *The Jacobian variety*

We shall work with a general curve $\mathscr{C}$ of genus 2, over a ground field K of characteristic not equal to 2, 3 or 5, which may be taken to have hyperelliptic form

$$\mathscr{C} : Y^2 = F(X) = f_6 X^6 + f_5 X^5 + f_4 X^4 + f_3 X^3 + f_2 X^2 + f_1 X + f_0 \qquad (1)$$

with $f_0, \ldots, f_6$ in K, $f_6 \neq 0$, and $\Delta(F) \neq 0$, where $\Delta(F)$ is the discriminant of F. In $\mathbb{F}_5$ there is, for example, the curve $Y^2 = X^5 - X$ which is not birationally equivalent to the above form.

1 Canonical form. We shall normally suppose that the characteristic¶ of the ground field is not 2 and consider curves $\mathcal{C}$ of genus 2 in the shape

$$\mathcal{C} : \quad Y^2 = F(X), \qquad (1.1.1)$$

where

$$F(X) = f_0 + f_1 X + \ldots + f_6 X^6 \in k[X] \qquad (1.1.2)$$

2. Set-up

Let k be a field of characteristic not equal to two, k^s a separable closure of k, and $f = \sum_{i=0}^{6} f_i X^i \in k[X]$ a separable polynomial with $f_6 \neq 0$. Denote by Ω the set of the six roots of f in k^s, so that $k(\Omega)$ is the splitting field of f over k in k^s. Let C be the smooth projective

But what is so special about characteristic 2?

In algebraically closed fields of characteristic 2, the 2-torsion of the Jacobian of a curve $\mathcal{C}$ of genus g is

$$\mathcal{J}(\mathcal{C})[2] \cong (\mathbb{Z}/2\mathbb{Z})^r$$

for some $0 \leq r \leq g$.

The singularities of Kummer surfaces

Characteristic	2			Not 2
2-rank	0	1	2	
Number of singularities	1	2	4	16
Singularity type	Elliptic	D_8	D_4	A_1

The desingularisations

For a Kummer surface defined over a field of characteristic different than 2 we know an explicit model of its desingularisation described as the intersection of 3 quadrics in $\mathbb{P}^5$.

But how can we obtain desingularised models of Kummer surfaces in characteristic 2?

Work in progress

For a general genus 2 curve $\mathcal{C} : y^2 + h(x)y = f(x)$ defined over a number field whose Jacobian has good reduction at all primes lying above 2, I am working on computing a basis of $\mathcal{L}(2(\Theta_+ + \Theta_-))$ which ”behaves well” when reducing modulo 2.

As a byproduct of this computation, models of partial desingularisations of Kummer surfaces in characteristic 2 can be found.

A final goal

Is it possible to construct explicit models of Kummer surfaces defined over a number field with everywhere good reduction?

Is this possible over quadratic fields?


Thank you!