$\Gamma(n), \Gamma_1(n)$ and $\Gamma_0(n)$-structures

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1 Introduction

The main reference for this talk is the book Arithmetic Moduli of Elliptic Curves [KM85] by Katz and Mazur and, more specifically, chapters 1 and 3.

2 Effective Cartier divisors

2.1 What is an effective Cartier divisors over a scheme?

An intuitive way of describing effective Cartier divisors on an S-scheme, is that they are families of effective divisors parametrised by S that behave in a sensible way. More rigurously,

Definition 2.1. *Let S be an arbitrary scheme and let X be an S-scheme (that means there is a proper smooth morphism $\pi : X \to S$). An* ***effective Cartier divisor*** *D in X/S is a closed subscheme $D \subset X$ such that*

- *D is flat over S, i.e. the restriction morphism $\pi|_S : D \to S$ is flat (intuitively, the fibers of the map are of constant dimension).*
- *The ideal sheaf $\mathcal{I}(D) \subset \mathcal{O}_X$ is an invertible $\mathcal{O}_X$-module i.e. it is a locally free $\mathcal{O}_X$-module of rank 1.*

It is important to remark that this notion is "local" on S. Namely, whenever S is an affine scheme $S = \operatorname{Spec}(R)$, we can cover X by affine opens $U_i = \operatorname{Spec}(A_i)$ where A_i is an R-algebra such that $D \cap U_i$ is defined in U_i by one equation $f_i = 0$, where $f_i \in A_i$ is an element such that

- A_i/f_iA_i is flat over R.
- f_i is not a zero divisor in A_i.

The ideal sheaf of D fits in an exact sequence

$$0 \longrightarrow \mathcal{I}(D) \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_D \longrightarrow 0.$$

In every open subset $U_i = \mathrm{Spec}(A_i)$, this becomes the exact sequence

$$0 \longrightarrow A_i \xrightarrow{\times f_i} A_i \longrightarrow A_i/f_iA_i \longrightarrow 0.$$

Given two effective Cartier divisors D and D' in X/S, their sum $D + D'$ is the effective Cartier divisor in X/S defined locally by the product of the defining equations of D and D'. Explicitly, if $S = \mathrm{Spec}(R)$ and if on an affine open set $\mathrm{Spec}(A)$ of X, D and D' are defined respectively by equations $f = 0$ and $g = 0$, then $D + D'$ is defined in that open set by $fg = 0$. For all $f, g \in A$, we then have a short exact sequence

$$0 \longrightarrow A/gA \xrightarrow{\times f} A/fgA \longrightarrow A/fA \longrightarrow 0.$$

Remark 2.2. *There is another interpretation of effective Cartier divisors as pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module and $s \in H^0(X, \mathcal{L})$ which sits in a short exact sequence of $\mathcal{O}_X$-modules*

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\times s} \mathcal{L} \longrightarrow \mathcal{L}/\mathcal{O}_X \longrightarrow 0$$

with $\mathcal{L}/\mathcal{O}_X$ flat over S.

2.2 Examples of effective Cartier divisors

1. Let k be a field and let X be a smooth variety over k. Then, in $X/\mathrm{Spec}(k)$, **effective Cartier divisors** are precisely in correspondence with **effective Weil divisors** (closed integral subschemes D of codimension one).

2. Let k be a field and let X be the affine smooth variety

$$X : xy - z^2 = 0 \subset \mathbb{A}^3$$

Then, the closed subscheme $D \subset X$ defined by

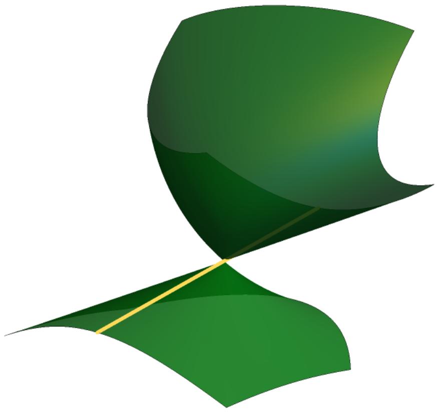
$$D : x = z = 0$$

is an effective Cartier divisor but not a Weil divisor (though $2D$ is a Weil divisor) [Har77, Example 6.11.13].

3. Let E be the curve over $\mathbb{A}^1_t \setminus \{-1, 8\}$ defined by

$$E : zy^2 = (x - 2z)(4x(x - z) - tz^2) \subset \mathbb{P}^2_{x,y,z} \tag{2.3}$$

Then, $D_t = \{[2 : 0 : 1]\}$ is an effective Cartier divisor. If we interpret E as an elliptic surface, D_t would be given by the yellow line in the following plot:



2.3 Effective Cartier divisors in curves

Let C/S be a smooth curve defined over a scheme S, which is a smooth morphism $C \to S$ of relative dimension one which is separated and of finite presentation. In a similar way as in the first example that has been given, there are nice characterisations for the Cartier divisors of C [KM85, Lemma 1.2.3.]. For instance, any section $s \in C(S)$ defines an effective Cartier divisor $[s]$ in C/S, as we have seen in the last example.

Given a proper effective Cartier divisor D over S (all effective Cartier divisors are proper if we assume the morphism $C \to S$ is proper, so this is not much to ask), we can always associate a notion of **degree** to it. The idea is that, locally on $S = \mathrm{Spec}(R)$, the affine ring of D is a locally free R-module of finite rank r, and as this is well-defined (it is constant Zariski locally on S), we can define $\deg(D)$ to be r.

Two nice facts about this notion of degree is that any effective Cartier divisor $[s]$ associated to a section $s \in C(s)$ is proper an has degree 1 and, conversely, every proper effective Cartier divisor of degree 1 is $[s]$ for some section $s \in C(s)$. Furthermore, if we have two proper effective Cartier divisors D_1 and D_2 in C/S we have that

$$\deg(D_1 + D_2) = \deg(D_1) + \deg(D_2).$$

3 *A*-structures on elliptic curves

Let E/S be a smooth commutative group-scheme over S of relative dimension one (the example that we must have in mind is when E is an elliptic curve over S) and let A be a finite abelian group.

Definition 3.1. *An* ***A-structure*** *on E/S is a homomorphism of abstract groups*

$$\phi : A \longrightarrow E(S)$$

such that the effective Cartier divisor D defined as

$$D = \sum_{a \in A} [\phi(a)] \qquad\qquad \deg(D) = \#A$$

is a subgroup G of E/S.

We call the subgroup G the **A-subgroup generated by** ϕ and we call ϕ an ***A*-generator** of G.

There is an alternative equivalent definition to A-structures that is sometimes easier to check, which is characterised by the following proposition [KM85, Lemma 1.5.3.].

Proposition 3.2. *Let A, E/S and ϕ as before. Then, ϕ is an A-structure on E/S if and only if for every geometric point* $\mathrm{Spec}(k) \to S$ *of S, the induced homomorphism*

$$\phi_k : A \longrightarrow E(k)$$

is injective.

Let's now see some examples of A-structures:

3.1 Γ(n)-structures

Let E/S be an elliptic curve over S, then

Definition 3.3. *A* **Γ(n)*-structure*** *on E/S is a $(\mathbb{Z}/n\mathbb{Z})^2$-structure on $E[n]/S$.*

Explicitly, this means that the effective Cartier divisor

$$\sum_{a,b \pmod{n}} [\phi(a,b)]$$

is a subgroup scheme of E.

An example of a $\Gamma[2]$-structure is the elliptic curve over $\mathrm{Spec}(\mathbb{Z}_{(2)})$ (the integers localised at the prime 2) defined by the equation

$$E : zy^2 = x(x-z)(x+z)$$

Then, ϕ is given by

$$\begin{aligned} \phi : (\mathbb{Z}/2\mathbb{Z})^2 &\longrightarrow E[\mathrm{Spec}(\mathbb{Z}_{(2)})] \\ (0,0) &\longmapsto [0:1:0] \\ (0,1) &\longmapsto [1:0:1] \\ (1,0) &\longmapsto [-1:0:1] \end{aligned}$$

and the effective Cartier divisor associated to this structure is

$$D = [0:1:0] + [1:0:1] + [-1:0:1] + [0:0:1]$$

3.2 $\mathbf{\Gamma_1(n)}$-structures

Let E/S be an elliptic curve over S, then

Definition 3.4. *A* $\mathbf{\Gamma_1(n)}$***-structure*** *on E/S is an $\mathbb{Z}/n\mathbb{Z}$-structure on $E[n]/S$.*

Explicitly, this means that the effective Cartier divisor

$$\sum_{a \pmod n} [\phi(a)]$$

is a subgroup scheme of E.

An example of a $\Gamma_1[2]$-structure is the elliptic curve defined by the equation 2.3 if the ground field has characteristic different than 2. Then, ϕ is given by

$$\begin{aligned} \phi : \mathbb{Z}/2\mathbb{Z} &\longrightarrow E[2](\mathbb{A}^1 \setminus \{-1, 8\}) \\ 0 &\longmapsto [0:1:0] \\ 1 &\longmapsto [2:0:1] \end{aligned}$$

and the effective Cartier divisor associated to this structure is $D = [0:1:0] + [2:0:1]$.

In a $\Gamma_1(n)$-structure, the image of 1 is what is known as a **point of order** n, so in this case $P = [2:0:1]$ is a point of order 2.

Equivalent, we can describe a $\Gamma_1(n)$-structure on E/S as an n-isogeny of elliptic curves over S, $\pi : E \to E'$ together with a generator of $\ker(\pi)$, which is a point

$$P \in \ker(\pi)(S) \subset E[n](S)$$

such that the corresponding homomorphism

$$\begin{aligned} \varphi : \mathbb{Z}/N\mathbb{Z} &\longrightarrow \ker(\pi) \\ a &\longmapsto aP \end{aligned}$$

generates $\ker(\pi)$.

This motivates the following definition:

3.3 Balanced $\mathbf{\Gamma_1(n)}$-structures

Definition 3.5. *A **balanced $\mathbf{\Gamma_1(n)}$-structure** on E/S is a choice of an n-isogeny $\pi : E \to E'$ and two points $P \in E$ and $P' \in E'$ such that*

$$E \underset{\hat{\pi}}{\overset{\pi}{\rightleftarrows}} E'$$

where $\hat{\pi}$ is the dual isogeny (so that $\hat{\pi} \circ \pi = [n]_E$ and $\pi \circ \hat{\pi} = [n]_{E'}$), $P \in \ker(\pi)(S)$ is a generator of $\ker(\pi)$ and $P' \in \ker(\hat{\pi})(S)$ is a generator of $\ker(\hat{\pi})$.

This is an intermediate structure that is finer than regular $\Gamma_1(n)$-structures but without containing as much information as $\Gamma(n)$-structures. Finally, we have the following:

3.4 $\mathbf{\Gamma_0(n)}$-structures

Let $T \to S$ a "nice" morphism of schemes and let X/S a scheme over S. Then, $X \times_S T$ is a scheme over T that we will denote by X_T.

Definition 3.6. *A **$\mathbf{\Gamma_0(n)}$-structure** on E/S is an n-isogeny $\pi : E \to E'$ which is cyclic in the sense that locally fppf on S, $\ker(\pi)$ admits a generator. This means that there exists some faithfully flat, locally of finite presentation morphism $T \to S$ and a point $P \in E_T(T)$ of order n on E_T/T which generates $\ker(\pi)_T$ in E_T such that we have an equality of Cartier divisors*

$$\ker(\pi)_T = \sum_{a=1}^{n} [aP].$$

3.5 Factorisation into prime powers

In some sense, when n is a compound number that decomposes as $n = ab$ where a and b are coprime, we can decompose $\Gamma(n)$-structures into $\Gamma(a)$ and $\Gamma(b)$-structures, and the same is true for the other A-structures that we have studied. More precisely, we have the following result [KM85, Lemma 3.5.1.].

Theorem 3.7. *Suppose that $n = ab$ with a and b relatively prime. Then, for any elliptic curve E/S, we have functorial isomorphisms*

$$\begin{aligned}\Gamma(n)\text{-}\operatorname{Str}(E/S) &\longrightarrow \Gamma(a)\text{-}\operatorname{Str}(E/S) \times \Gamma(b)\text{-}\operatorname{Str}(E/S)\\ \Gamma_1(n)\text{-}\operatorname{Str}(E/S) &\longrightarrow \Gamma_1(a)\text{-}\operatorname{Str}(E/S) \times \Gamma_1(b)\text{-}\operatorname{Str}(E/S)\\ \operatorname{Bal}\Gamma(n)\text{-}\operatorname{Str}(E/S) &\longrightarrow \operatorname{Bal}\Gamma(a)\text{-}\operatorname{Str}(E/S) \times \operatorname{Bal}\Gamma(b)\text{-}\operatorname{Str}(E/S)\\ \Gamma_0(n)\text{-}\operatorname{Str}(E/S) &\longrightarrow \Gamma_0(a)\text{-}\operatorname{Str}(E/S) \times \Gamma_0(b)\text{-}\operatorname{Str}(E/S)\end{aligned}$$

In future talks, this functorial correspondence will be further explored when we study the representability of all of these A-structures.

References

[Har77] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer New York, New York, NY, 1977.

[KM85] Nicholas M. Katz and Barry Mazur. *Arithmetic Moduli of Elliptic Curves*. Princeton University Press, 12 1985.