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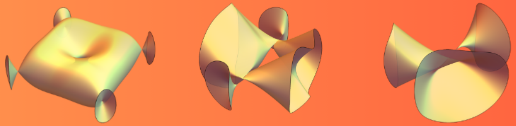
How to desingularise Kummer surfaces



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How to desingularise Kummer surfaces

A four-step guide on how to work with world's most singular quartic surfaces

Written by Alvaro Gonzalez Hernandez
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wikiHow to desingularise Kummer surfaces

1 Learn the definition and history of Kummer surfaces

The history of Kummer surfaces

1822

Fresnel discovered that the behaviour of light passing through a crystal can be modelled from the solutions of an equation of degree 4 in 3 variables.

1837

Hamilton found that the variety that defined that equation had 16 singular points.

1848

Cayley studied quartic surfaces with 16 nodes and discovered common properties to all of these.

And then, finally...

Kummer showed that, through coordinate changes, any quartic surface with 16 nodes could be described as a member of:

$$(x^2 + y^2 + z^2 + w^2 + A(xy + zw) + B(xz + yw) + C(xw + yz))^2 + Kxyzw = 0$$

where

$$K = a^2 + b^2 + c^2 - 2abc - 1$$

A provisional definition

Kummer surface (Definition I)

A **Kummer surface** is a quartic surface in $\mathbb{P}^3$ with 16 isolated singularities.

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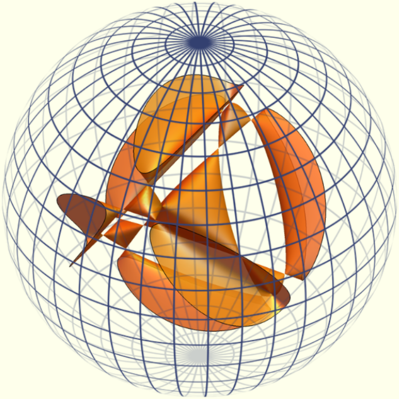
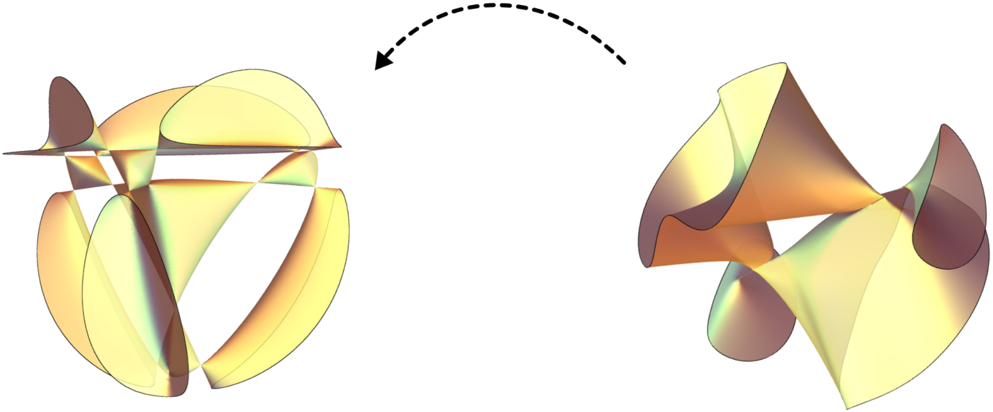
Question:

Is this a good definition by today's standards?

$\mathbb{P}^5$





Is this still a Kummer surface?

A better definition may then be

Kummer surface (Definition II)

A **Kummer surface** is a surface which is birationally equivalent to a quartic surface in $\mathbb{P}^3$ with 16 isolated singularities.

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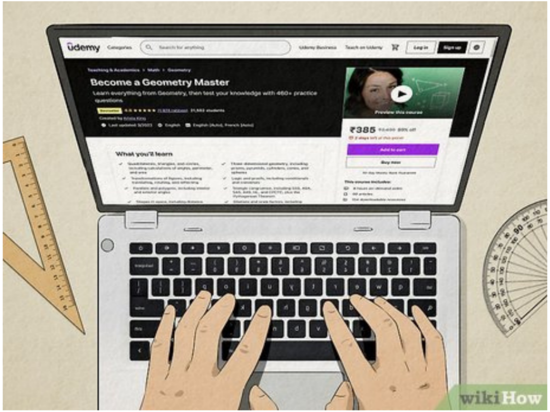
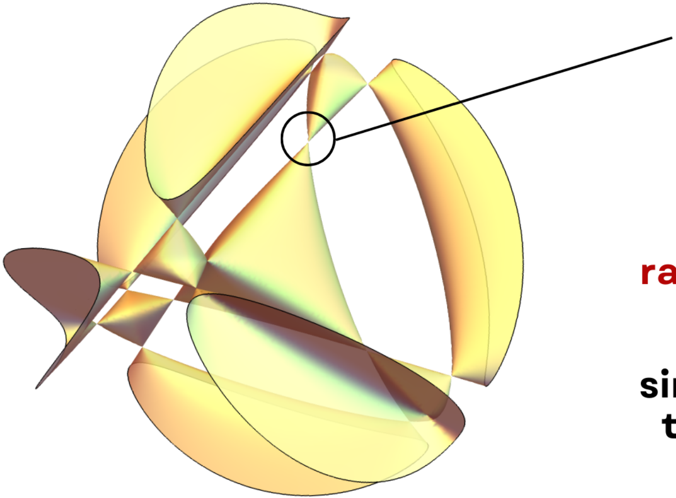


2 Become a master in blowing up surfaces

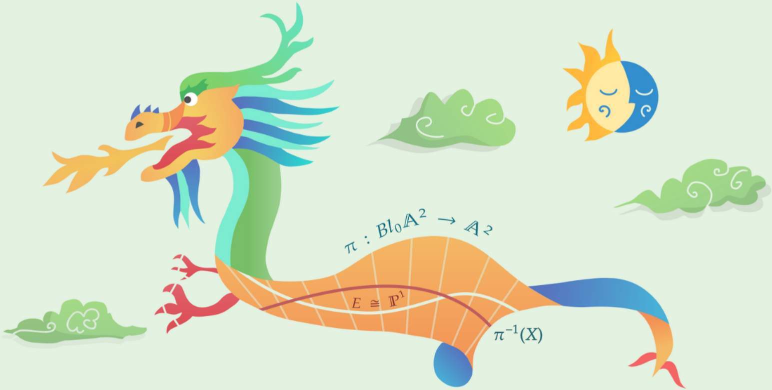


These are **nodes**, also known as **A_1 singularities.**

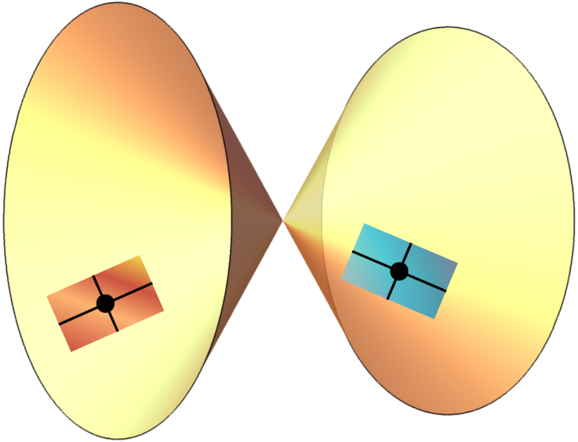
They are **du Val singularities** or **rational double points.**

Locally these singularities resemble the vertex of a cone

Given a singular surface, we can always find a birationally equivalent smooth surface by applying a series of transformation known as blow-ups.



Classical example of a blow-up



Blow-up of a cone

Let C be the **cone** described by the equation $x_2^2 - x_1x_3 = 0$ in $\mathbb{P}^3$, which is singular at $[0:0:0:1]$ and let $\tilde{C}$ be the surface in $\mathbb{P}^3 \times \mathbb{P}^2$ defined by the equations:

$$\begin{aligned} x_2^2 - x_1x_3 &= 0 \\ y_2^2 - y_1y_3 &= 0 \\ x_1y_2 - x_2y_1 &= 0 \\ x_1y_3 - x_3y_1 &= 0 \\ x_2y_3 - x_3y_2 &= 0 \end{aligned}$$

Then, the obvious map $\pi : \tilde{C} \to C$ blows up the point $[0:0:0:1]$.

We have motivated why, given a surface with isolated singularities, it is always possible to find a smooth surface that is birationally equivalent to it through blow-ups.

Let X be the desingularisation of a Kummer surface. Then, using multiple tools from our cohomology toolkit we can deduce many properties about the geometry of X. For instance, that X is what is known as a **K3 surface**.

K3 surfaces

A **K3 surface** X is a simply connected surface with trivial canonical bundle, meaning

$$h^1(X, \mathcal{O}_X) = 0,$$
$$\omega_X = \wedge^2 \Omega_X \simeq \mathcal{O}_X.$$

They have interesting geometric properties due to the fact that they are the **Calabi-Yau varieties** of dimension 2.

Abelian surfaces

From the Enriques-Kodaira classification of surfaces, we know that there is only one other type of surface with trivial canonical bundle. These are known as **Abelian surfaces** and they are also very special.

Abelian surfaces

An **Abelian surface** is a complete group variety of dimension 2.

Basically, Abelian surfaces are algebraic surfaces that have a **geometric group law** that allows us to add points in it.

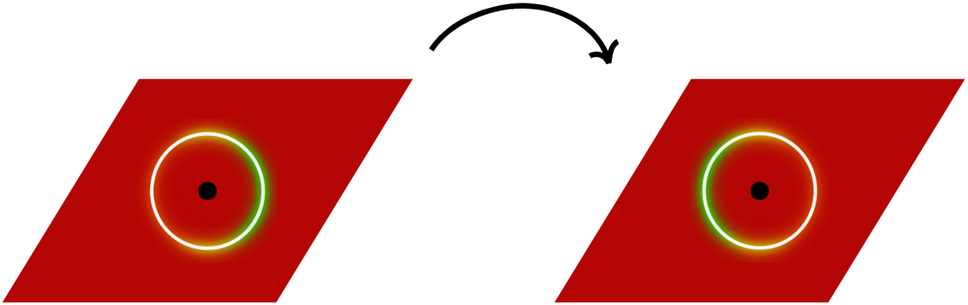
Going back to our history lesson

While investigating how to embed Abelian varieties in projective space, Göpel found that some the theta functions that were involved in this embedding defined the equation of a Kummer surface.

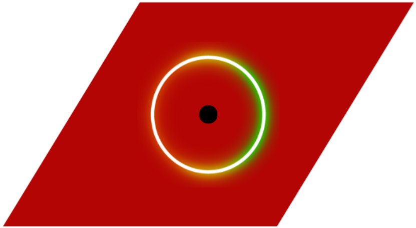
But why?

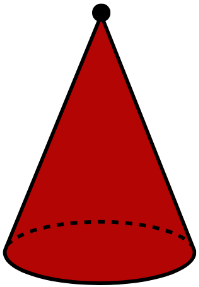
Suppose that we have a finite group acting on a variety

$\mathbb{Z}/2\mathbb{Z}$ acting on the plane by a 180° rotation



We can then create a quotient variety by identifying every point with its image under the action





This quotient variety satisfies:

1- Outside of the fixed points of the action, the geometry of the quotient variety is close to the original variety.

2- The fixed points of the action generally become singular in the quotient variety.

3- Starting with an embedding of our variety in projective space we can compute an embedding of the quotient variety by considering global sections that are invariant under the action.

Abelian varieties have a natural $\mathbb{Z}/2\mathbb{Z}$ action which consists of sending any point on the variety to its inverse with respect to the group law!

The **fixed points** of the action are the points that are equal to its inverse, therefore, they are the identity and the points of order 2 (the **2-torsion of the group**).

Over a field of characteristic, not 2, the 2-torsion of an Abelian surface turns out to be

$$(\mathbb{Z}/2\mathbb{Z})^4$$

While investigating how to embed Abelian varieties in projective space, Göpel found that some the theta functions that were involved in this embedding defined the equation of a Kummer surface.

Why?

He had found 4 theta functions that were invariant under the $\mathbb{Z}/2\mathbb{Z}$ action, and therefore generated the quotient of the Abelian surface.

Kummer surface (Definition III)

Let $\mathcal{A}$ be an Abelian surface and let ι be the involution in $\mathcal{A}$ that sends an element to its inverse. Then, the **Kummer surface** associated to $\mathcal{A}$, $\mathrm{Kum}(\mathcal{A})$ is the quotient variety $\mathcal{A}/\iota$.

This definition has the advantage that it can easily be generalised to other dimensions.

Also, it allow us to study Kummer surfaces from the properties of Abelian surfaces.

How to desingularise Kummer surfaces

Idea:

Starting with an embedding of a variety in projective space, we can compute an embedding of its quotient variety by studying the functions that are invariant under the action.

If we are studying a $\mathbb{Z}/2\mathbb{Z}$ action, we can also study the sections that change sign when we apply the action.

Action(even function) = even function
Action(odd function) = -odd function

Desingularising a quotient variety: the cone

Consider $\mathbb{P}^2$, whose points are of the form $[x_1 : x_2 : x_3]$ and consider the $\mathbb{Z}/2\mathbb{Z}$ action ι given by

$$\iota : \begin{cases} x_1 \mapsto -x_1 \\ x_2 \mapsto -x_2 \\ x_3 \mapsto x_3 \end{cases}$$

What are the even quadratic functions on the variables $\{x_1, x_2, x_3\}$?

Desingularising a quotient variety: the cone

Consider $\mathbb{P}^2$, whose points are of the form $[x_1 : x_2 : x_3]$ and consider the $\mathbb{Z}/2\mathbb{Z}$ action ι given by

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What are the even quadratic functions on the variables $\{x_1, x_2, x_3\}$?

$$e_1 = x_1^2 \qquad e_2 = x_1 x_2 \qquad e_3 = x_2^2 \qquad e_4 = x_3^2$$

Desingularising a quotient variety: the cone

The quotient variety $\mathbb{P}^2/\iota$ can be embedded in $\mathbb{P}^3$ by considering the points given by

$$[e_1 : e_2 : e_3 : e_4] = [x_1^2 : x_1x_2 : x_2^2 : x_3^2].$$

From this description it is easy to check that there exist only a relation between $\{e_1, e_2, e_3, e_4\}$, which defines the equation of $C = \mathbb{P}^2/\iota$ inside of $\mathbb{P}^3$ as

$$e_2^2 - e_1e_3 = 0.$$

Desingularising a quotient variety: the cone

Now, let's see how we can desingularise this cone. Consider a basis of all cubic polynomials on x_1, x_2, x_3 that are odd with respect to ι. Then, in a similar fashion as before, the map given by

$$[o_1 : o_2 : o_3 : o_4 : o_5 : o_6] = [x_1^3 : x_1^2 x_2 : x_1 x_2^2 : x_2^3 : x_1 x_3^2 : x_2 x_3^2]$$

gives an embedding of $\tilde{C} = \mathbb{P}^2/\iota$ inside of $\mathbb{P}^5$, given by the equations:

$$\begin{aligned} o_1 o_3 - o_2^2 &= 0 & \qquad o_2 o_4 - o_3^2 &= 0 \\ o_1 o_4 - o_2 o_3 &= 0 & \qquad o_1 o_6 - o_2 o_5 &= 0 \\ o_2 o_6 - o_3 o_5 &= 0 & \qquad o_3 o_6 - o_4 o_5 &= 0 \end{aligned}$$

Desingularising a quotient variety: the cone

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As

$$[e_1 : e_2 : e_3 : e_4] = [x_1^2 : x_1x_2 : x_2^2 : x_3^2]$$
$$[o_1 : o_2 : o_3 : o_4 : o_5 : o_6] = [x_1^3 : x_1^2x_2 : x_1x_2^2 : x_2^3 : x_1x_3^2 : x_2x_3^2],$$

it is easy to see that we can define a birational map $\pi : \tilde{C} \to C$

$$\pi([o_1 : o_2 : o_3 : o_4 : o_5 : o_6]) = \begin{cases} [o_1 : o_2 : o_3 : o_5] \text{ if } o_1 \neq 0 \\ [o_2 : o_3 : o_4 : o_6] \text{ if } o_2 \neq 0 \end{cases}$$

Desingularising a quotient variety: the cone

The map $\pi : \tilde{C} \to C$

$$\pi([o_1 : o_2 : o_3 : o_4 : o_5 : o_6]) = \begin{cases} [o_1 : o_2 : o_3 : o_5] & \text{if } o_1 \neq 0 \\ [o_2 : o_3 : o_4 : o_6] & \text{if } o_2 \neq 0 \end{cases}$$

is a blow-up of the point $[0 : 0 : 0 : 1]$.

It also contracts the exceptional divisor $E \subset \tilde{C}$ which is given by

$$o_1 o_3 - o_2^2 = 0$$
$$o_1 o_4 - o_2 o_3 = 0$$

Going back to our Kummer surfaces

The procedure to find a desingularised model of a Kummer surface is essentially the same as the cone, with the ”small” differences that:

- Instead of being generated by only 3 global sections, as $\mathbb{P}^2$, general Abelian surfaces have been described to embed in $\mathbb{P}^{15}$ via 16 sections $\{x_1, \ldots, x_{16}\}$.
- The space of even sections is generated by 4 sections $\{e_1, \ldots, e_4\}$, that satisfy the quartic relation that describes a singular Kummer surface.

- The space of odd sections is generated by 6 sections $\{o_1, \ldots, o_6\}$, that satisfy 3 quadratic relations. Therefore, a projective model of a desingularised Kummer surface can be achieved as a complete intersection of 3 quadrics in $\mathbb{P}^5$.

Example of a Kummer surface

$$5e_2^4 - 10e_1e_2^2e_3 + 5e_1^2e_3^2 + 4e_1^3e_4 + 5e_3^3e_4 - e_2^2e_4^2 + 4e_1e_3e_4^2 = 0$$

Example of a desingularised Kummer surface

$$\begin{aligned} -10o_1o_2 + 10o_4o_5 + o_3o_6 &= 0 \\ -5o_1^2 + 25o_4^2 + 5o_3o_5 + o_2o_6 &= 0 \\ 25o_3o_4 + 5o_2o_5 + o_1o_6 &= 0 \end{aligned}$$

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3 **Put your skills to use by helping a number theorist**

Number theory (or **arithmetic** or **higher arithmetic** in older usage) is a branch of pure mathematics devoted primarily to the study of the integers and arithmetic functions. German mathematician Carl Friedrich Gauss (1777–1855) said, "Mathematics is the queen of the sciences—and number theory is the queen of mathematics."[1] Number theorists study prime numbers as well as the properties of mathematical objects constructed from integers (for example, rational numbers), or defined as generalizations of the integers (for example, algebraic integers).

$$+$$

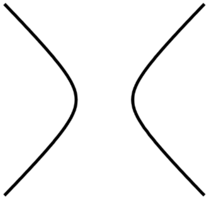
Algebraic geometry is a branch of mathematics which uses abstract algebraic techniques, mainly from commutative algebra, to solve geometrical problems. Classically, it studies zeros of multivariate polynomials; the modern approach generalizes this in a few different aspects.

$$=$$

Finding the set of rational points of a variety, i.e.
all rational solutions to a system of polynomial equations

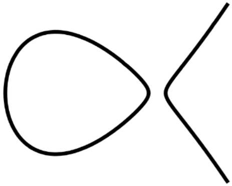
Classification of curves

Genus 0 curves



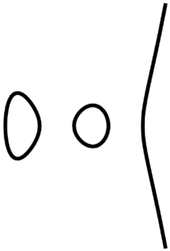
They can only have either no rational points or an infinite number of them

Genus 1 curves



They can have no rational points, a finite or an infinite number of them

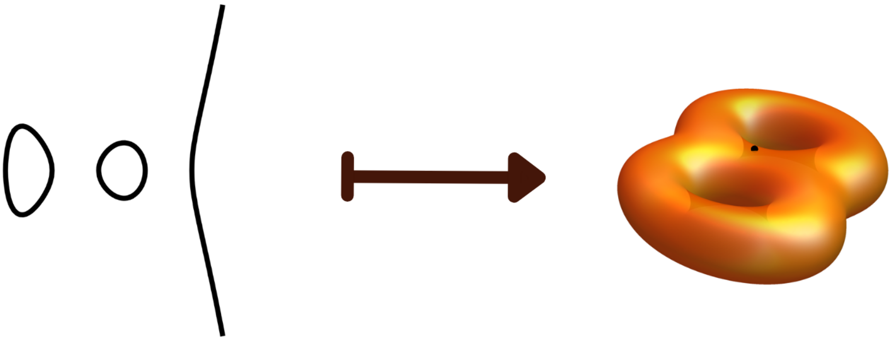
Higher genus curves



They can only have either no rational points or a finite number of them

Faltings (1983)

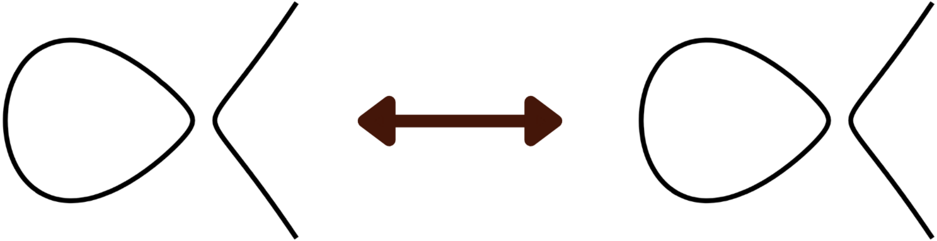
Higher genus curves



Genus g>1
(hyperelliptic) curves

Jacobian variety
of dimension g

In the case of elliptic curves



Elliptic curve **Jacobian variety**

Higher genus curves

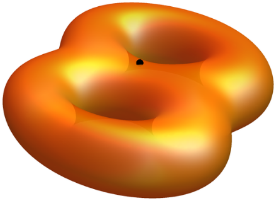
The Jacobian of a curve is an Abelian variety.

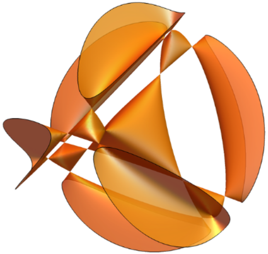
Mordell-Weil Theorem (II)

The set of rational points of an Abelian variety is a finitely generated Abelian group. Therefore, if $\mathrm{Jac}(\mathcal{C})$ is the Jacobian variety associated to a curve $\mathcal{C}$:

$$\mathrm{Jac}(\mathcal{C})(\mathbb{Q}) \cong \mathrm{Jac}(\mathcal{C})(\mathbb{Q})_{\text{torsion}} \times \mathbb{Z}^r \qquad \text{for some } r \geq 0.$$

How does knowing about Kummer surfaces help to compute rational points on curves?



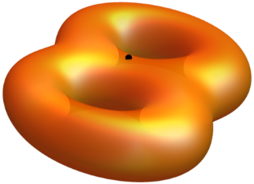


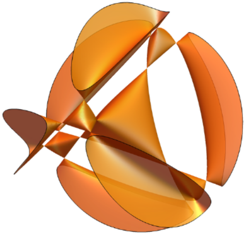
For a general genus 2 curve, defining the equations of the Jacobian is really complicated...

...but the equation of the Kummer is really easy to find

Solution:

Find rational points in the Kummer and lift them to the Jacobian





This involves computing the odd functions in the Jacobian with respect to the action (which also defined our desingularisation)!

4 **Set yourself a challenge, try to do it now in characteristic 2!**

1 Canonical form. We shall normally suppose that the characteristic¶ of the ground field is not 2 and consider curves $\mathcal{C}$ of genus 2 in the shape

$$\mathcal{C}: \quad Y^2 = F(X), \tag{1.1.1}$$

where

$$F(X) = f_0 + f_1 X + \ldots + f_6 X^6 \in k[X] \tag{1.1.2}$$

1. *The Jacobian variety*

We shall work with a general curve $\mathscr{C}$ of genus 2, over a ground field K of characteristic not equal to 2, 3 or 5, which may be taken to have hyperelliptic form

$$\mathscr{C} : Y^2 = F(X) = f_6X^6 + f_5X^5 + f_4X^4 + f_3X^3 + f_2X^2 + f_1X + f_0 \tag{1}$$

with $f_0, \ldots, f_6$ in K, $f_6 \neq 0$, and $\Delta(F) \neq 0$, where $\Delta(F)$ is the discriminant of F. In $\mathbb{F}_5$ there is, for example, the curve $Y^2 = X^5 - X$ which is not birationally equivalent to the above form.

1 Canonical form. We shall normally suppose that the characteristic¶ of the ground field is not 2 and consider curves $\mathcal{C}$ of genus 2 in the shape

$$\mathcal{C} : \quad Y^2 = F(X), \tag{1.1.1}$$

where

$$F(X) = f_0 + f_1X + \ldots + f_6X^6 \in k[X] \tag{1.1.2}$$

2. Set-up

Let k be a field of characteristic not equal to two, k^s a separable closure of k, and $f = \sum_{i=0}^{6} f_iX^i \in k[X]$ a separable polynomial with $f_6 \neq 0$. Denote by Ω the set of the six roots of f in k^s, so that $k(\Omega)$ is the splitting field of f over k in k^s. Let C be the smooth projective

One interesting thing about characteristic 2 is:

There are no minus signs!

Therefore, separating functions into even and odd no longer makes sense.

So what can we say about Kummer surfaces over fields of characteristic 2?

But, what is so special about characteristic 2?

Fact

For $g > 1$, the set of points of order at most 2 is the set of all fixed points under the action of ι and these points are singular points of our Kummer surface.

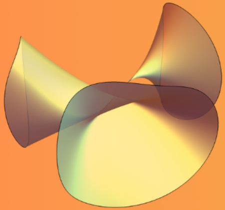
In algebraically closed fields of characteristic 2, the 2-torsion of an Abelian variety $\mathcal{A}$ of dimension g is

$$\mathcal{A}[2] \cong (\mathbb{Z}/2\mathbb{Z})^r$$

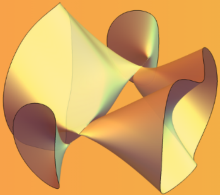
for some $0 \leq r \leq g$.

Characteristic	2			Not 2
2-rank	0	1	2	
Number of singularities	1	2	4	16
Singularity type	Elliptic	D_8	D_4	A_1

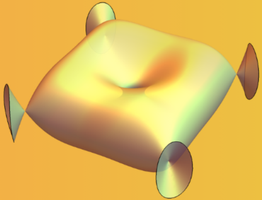
Characteristic 2



Supersingular

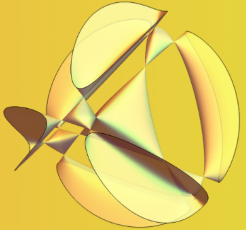


“Almost” Ordinary



Ordinary

Characteristic different than 2



Kummer surfaces in characteristic 2

Arithmetic side

Geometric side

Theorem / Computation (G.)

Given a genus 2 curve $\mathcal{C}$ defined over a field k of characteristic 2, it is possible to find a basis that gives an explicit embedding of $\mathrm{Jac}(\mathcal{C})$ inside of $\mathbb{P}^{15}$.

$\Rightarrow$ With small modifications, we can repeat the reasoning of the previous example to study curves over fields of characteristic 2.

Geometric side

Theorem (G.)

Using the previously computed basis, it is possible to compute embeddings of partial desingularisations of Kummer surfaces in characteristic 2.

2-rank	0	1	2
Number of tropes	1	2	4
Singularities	$1\times$ Elliptic	$2 \times D_8$	$4 \times D_4$
Singularities after partial desingularisation	$1\times$ Simpler elliptic	$2 \times D_4 + 2 \times A_3$	$12 \times A_1$

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