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X_i real valued iid r.v.

$\varphi(t) = \mathbb{E}[e^{tX}] < \infty$

$\mathbb{E}(X_i) = m \qquad x > m$

$\frac{1}{n} \log \mathbb{P}\left[\frac{1}{n}\sum_{i=1}^{n} X_i \geq x\right] \xrightarrow{n\to\infty} -I(x)$

$I(x) = \sup_{t\in\mathbb{R}} \left(tx - \log \varphi(t)\right)$

$X_i \sim \mu$

$I(x)$ was given as $H(\nu_{t} \| \mu)$, $t = t(x)$ $\qquad \nu_t$ tilted meas.

Chapter 2 Large deviation principles

(E, d) metric space (separable)

μ_n proba. meas. on E

$\gamma_n \in \mathbb{R}_+ \uparrow \infty \quad n \to \infty$ (scale)

$I : E \longrightarrow [0, \infty]$ rate function.

Def: μ_n satisfies a Large deviation principle (LDP) if

(i) I has compact sublevelsets (i.e. $\forall R > 0$, $\{x \in E : I(x) \leq R\}$ compact)

(ii) $\forall \mathcal{O} \subset E$ open

$$\liminf \frac{1}{\gamma_n} \log \mu_n(\mathcal{O}) \geq -\inf_{x\in\mathcal{O}} I(x)$$

(iii) $\forall \mathcal{C} \subset E$ closed

$$\limsup_{n\to\infty} \frac{1}{\gamma_n} \log \mu_n(\mathcal{C}) \leq -\inf_{x\in\mathcal{C}} I(x)$$

Remarks

- If I has a unique zero ($\exists! \, x \; I(x) = 0$) and $\gamma_n \uparrow \infty$ quickly enough $\Rightarrow$ if $X_n \sim \mu_n$ then $X_n \to x$ a.s.
- $\exists$ at least one x, $I(x) = 0$.

- **Lemma:** Rate fct is unique. If μ_n satisfies LDP with rate fct. I and J then $I(x) = J(x) \; \forall x$.

proof: Fix $x \in E$

$$
\begin{aligned}
-I(x) &\leq -\inf_{y\in B_{1/N}(x)} I(y) \\
&\leq \liminf \frac{1}{\gamma_n} \log \mu_n\left(B_{1/N}(x)\right) \\
&\leq \limsup \frac{1}{\gamma_n} \log \mu_n\left(\overline{B}_{1/N}(x)\right) \\
&\leq -\inf_{y\in\overline{B}_{1/N}} J(y)
\end{aligned}
$$

By l.s.c. $\Rightarrow I(x) \leq -J(x)$ $\square$

· Idea: $\mu_n(A) \approx \exp\left(-\gamma_n \left(\inf_{y \in A} I(y) + o(1)\right)\right)$

This cannot be true for every set A.

· Topology is important!!!

· Recall: μ_n proba. measures on E converges weakly to μ if

$\forall \mathcal{O}$ open $\liminf_{n\to\infty} \mu_n(\mathcal{O}) \geq \mu(\mathcal{O})$

$\forall \mathcal{C}$ $\limsup_{n\to\infty} \mu_n(\mathcal{C}) \leq \mu(\mathcal{C})$

LDP "weak convergence on exponential scale"

$\mu_n \to \mu$ weakly iff $\forall f: E \to \mathbb{R}$ cont. bdd

$$\int f(x)\mu_n(dx) \to \int f(x)\mu(dx)$$

there is an exponential version of this that characterizes LDP.

· Definition: μ_n proba. meas. on E are exponentially tight iff $\forall R > 0$ $\exists K_R$ compact such that

$$\limsup_{n\to\infty} \frac{1}{\gamma_n} \log \mu_n(K_R^c) \leq -R.$$

Exo: μ_n satisfies LDP $\Rightarrow$ exponentially tight

not Ex: μ_n exponentially tight $\Rightarrow \exists$ subsequence μ_{k_n} that satisfies LDP.

* Recall: μ_n are tight if $\forall \varepsilon > 0$ K_ε compact

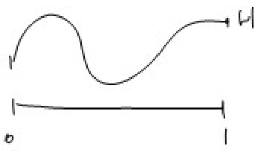
$\mu_n(K_\varepsilon^c) \leq \varepsilon$

$\mu_n \to \mu$ implies tightness, tightness implies weak convergence of a subseq.

Examples " Gibbs measure with decreasing temperature"

μ_n measures on $[0,1]$

$$\mu_n(dx) = \frac{1}{Z_n} \exp(-n H(x))$$



$\sim \exp(-nH(x))\,dx$

<u>Lemma</u> (Say H continuous). Then μ_n satisfy a LDP with rate fct.

$$H(x) - \inf_{y \in [0,1]} H(y)$$

<u>Remark</u> Laplace method for det. limiting behavior of exponential integrals.

<u>Proof</u>: Lower bound on open sets:

$\mathcal{O}$ open, $x \in \overline{\mathcal{O}}$, $\inf_{y \in \mathcal{O}} I(y) = I(x)$

By continuity in a ball of radius δ around x

$H(y) \leq H(x) + \delta$

$$\int_{\mathcal{O}} \exp(-nH(z))\,dz$$

$$\geq \int_{\mathcal{O} \cap B_\delta(x)} \exp(-nH(z))\,dz$$

$$\geq \exp(-n(H(x)+\delta))\,\underbrace{\mathrm{Leb}(\mathcal{O} \cap B_\delta(x))}_{\geq c_\delta}$$

$$\overline{\lim}\ \frac{1}{n} \log \int_{\mathcal{O}} \exp(-nH(z))\,dz \geq -(H(x)+\delta)$$

δ arbitrary $\to 0$!

<u>Upper bound</u>:
$$\int_{\mathcal{C}} e^{-nH(z)}\,dz \leq e^{-n \inf_{\mathcal{C}} H} \int 1\,dz \leq e^{-n \inf_{\mathcal{C}} H}\, C_H \quad . \square$$

2.1 <u>An infinite dimensional example</u>

<u>Brownian motion</u>.

- B_t $t \in [0,1]$, standard one-dim. BM.

<u>recall</u>: this means that $(B_t, t \in [0,1])$ Gaussian centred process, with

$\circledast$ · $\mathbb{E}(B_t B_s) = t \wedge s \quad \forall s$

· $t \mapsto B_t$ is continuous a.s.

$\circledast$ can be rewritten as $\forall t_1 < t_2 < \dots < t_n$

$B_{t_i} - B_{t_{i-1}}$ ind. and Gaussian, centred

with $\mathbb{E}\left((B_{t_i} - B_{t_{i-1}})^2\right) = t_i - t_{i-1}$.

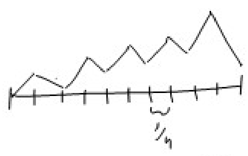
<u>Question</u> $\sqrt{\varepsilon}\, B_t \longrightarrow 0$ a.s. as $\varepsilon \to 0$.

How does $\mathbb{P}[\sqrt{\varepsilon}\, B_t \in A]$ decay.

In particular, we want an LDP on
$$E = C([0,1])$$

<u>Heuristic derivation:</u>

$$B_{\frac{k}{n}} - B_{\frac{k-1}{n}} \sim N(0, \tfrac{1}{n})$$



$$B^n \sim \frac{1}{Z^n} \exp\left(\sum_{k=1}^{n} \frac{\left(B_{\frac{k}{n}} - B_{\frac{k-1}{n}}\right)^2}{2\frac{1}{n}} \right) d\mathcal{L}^n$$

$$\frac{1}{2} \sum \left(\frac{B_{\frac{k}{n}} - B_{\frac{k-1}{n}}}{\frac{1}{n}} \right)^2 \frac{1}{n}$$

$$\longrightarrow \frac{1}{2} \int_0^1 \dot{B}_t^2 \, dt$$

distribution $\sim \frac{1}{Z} \exp\left(-\frac{1}{2}\int_0^1 \dot{B}_t^2 \, dt\right) \text{``}d\mathcal{L}\text{''}$

This is <u>wrong</u>! None of these terms make sense!!

If it were true distribution of $\sqrt{\varepsilon} B_t$

$$\sim \frac{1}{Z} \exp\left(-\frac{1}{2\varepsilon} \int \dot{B}_t^2 \, dt\right) d\mathcal{L}$$

the guess is

$\sqrt{\varepsilon} B_t$ satisfy an LDP with rate function $\frac{1}{2} \int_0^1 \dot{B}_t^2 \, dt$.

<u>2nd Heuristic derivation:</u> $\underbrace{B_t^1 + B_t^2}_{\text{ind. BM.}} \overset{\mathcal{L}}{=} \sqrt{2}\, B_t$

$$\frac{1}{n} \sum_{i=1}^{n} B_t^i \overset{\text{law}}{=} \frac{1}{\sqrt{n}} B_t$$

<u>Exo</u> (Deuschel-Stroock) Check that the Legendre transform of log-moment gen. fct. gives you the same answer.

<u>Def</u>: (Cameron-Martin space). We call H the set of all continuous fct. f that have the property that $\forall s \in [0,1]$
$f(s) = \int_0^s g(t)\,dt$ (f is a.c.) and s.t. that

$$\|f\|_H^2 = \int_0^1 g(t)^2 \, dt < \infty.$$

<u>Remark</u>: This is (essentially) the Sobolev space H^1, $W^{1,2}$.

We define the rate fct. $I : E = C([0,1]) \to \mathbb{R}_+ \cup \{+\infty\}$

$$I(x) = \begin{cases} \frac{1}{2}\|x\|_H^2 & x \in H \\ +\infty & \text{else} \end{cases}$$

<u>Thm</u> (Schilder '66)

$(\sqrt{\varepsilon}\, B_t)$ satisfy a LDP with rate fct. I.

<u>proof</u>: ① Compactness of Sublevelsets

We will prove for upper and lower bounds the following

② $\forall x, \forall \delta > 0$

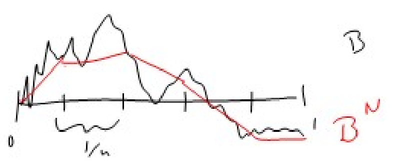
$$\liminf \varepsilon \log \mathbb{P}\left[\sqrt{\varepsilon} B_\cdot \in B_\delta(x) \right] \geq -I(x)$$

③ $\forall R > 0, \forall \delta$, $\Phi(R) = \{x : I(x) \leq R\}$

$$\overline{\lim}\ \varepsilon \log \mathbb{P}\left[\underset{\geq \delta}{\operatorname{dist}(\Phi(R), \sqrt{\varepsilon} B_\cdot)} \right] \leq -R.$$

Let's start with

③ Fix $n \in \mathbb{N}_0$, Let B^n be the piecewise linearised Brownian motion.



$$\mathbb{P}\left[\operatorname{dist}_{L^\infty}(\sqrt{\varepsilon} B, \Phi(\delta)) > \delta \right]$$

$$\leq \mathbb{P}\left[\|\sqrt{\varepsilon} B - \sqrt{\varepsilon} B_N\|_\infty \geq \delta \right] \quad ①$$

$$+ \mathbb{P}\left[I(B_N) \geq R \right]. \quad ②$$

$$① \leq \sum_{k=0}^{n-1} \mathbb{P}\left[\sup_{[\frac{k}{n}, \frac{k+1}{n})} |B - B_N| \geq \frac{\delta}{\sqrt{\varepsilon}} \right]$$

$$= n\, \mathbb{P}\left[\sup_{t \in [0, \frac{1}{n})} \left| B_t - \frac{n}{t} B_{1/n} \right| \geq \frac{\delta}{\sqrt{\varepsilon}} \right]$$

$$\leq n\, \mathbb{P}\left(\sup_{t \in [0, \frac{1}{n})} |B_t| \geq \frac{\delta}{2\sqrt{\varepsilon}} \right]$$

$$= n\, \mathbb{P}\left[\sup_{0 \leq t \leq 1} |B_t| \geq \frac{n^2 \delta}{2\sqrt{\varepsilon}} \right]$$

$$\leq n\, \mathbb{P}\left[\exp\left(\sup_{0 \leq t \leq 1} |B_t| \right) \geq \exp\left(\frac{n^2 \delta}{2\sqrt{\varepsilon}} \right) \right]$$

should be ε next time!

$$\leq n \exp\left(-\frac{n^2 \delta}{2\varepsilon} \right) \underbrace{\mathbb{E}\left(\exp\left(\sup_{0 \leq t \leq 1} |B_t| \right) \right)}_{< \infty}$$

$$\Rightarrow \varepsilon \log - {}^{n} - \quad \leq -\frac{n^2 \delta}{2} \leq -R$$

if n large enough!

I have used that

$$\mathbb{E}\left(\exp\left(\sup_{0 \leq s \leq 1} |B_s| \right) \right) < \infty$$

$\exp\left(B_t - \frac{1}{2} t \right)$ martingale.

maximal inequality: