Extra Lesson
Thursday 2pm

Last time

X_t Markov chain $\mathbb{Z}^d / N\mathbb{Z}^d$

empirical measures

$$\mu_t(A) = \frac{1}{t} \int_0^t \mathbb{1}_A (X_s)\, ds$$

Aim: μ_t satisfy LDP on $\mathcal{M}_1(\mathbb{Z}^d/N\mathbb{Z}^d)$

$$I(\nu) = \langle \Delta_N \sqrt{\nu}, \sqrt{\nu} \rangle$$

- Asymptotics of $\mathbb{E}\left(\exp\left(\int_0^t V(X_s)\, ds \right) \right)$ behaves like $e^{\lambda(V) t}$ where
  - $\lambda(V)$ largest Eigenvalue of $\Delta_N + V$.
- C^1 OK by implicit function thm.

$\Rightarrow$ LDP rate fct.

$$\bar{I}(\mu) = \sup_{V \in C_b} \left(\langle \mu, V \rangle - \lambda(V) \right)$$

$$\lambda(V) = \sup_{\substack{\varphi \in \ell^2 \\ \|\varphi\|^2 = 1}} \langle \varphi, \Delta_N \varphi \rangle + \langle V, \varphi^2 \rangle$$

$$\overset{\nu = \varphi^2}{=} \sup_{\nu \in \mathcal{M}_1} \langle \sqrt{\nu}, \Delta_N \sqrt{\nu} \rangle + \langle V, \nu \rangle$$

$=$ Legendre trafo of $\nu \mapsto -\langle \sqrt{\nu}, \Delta_N \sqrt{\nu} \rangle$ extended by $+\infty$ to $\mathcal{M}(\mathbb{Z}^d/N\mathbb{Z}^d)$

Convex + l.s.c. $\Rightarrow$ OK. $\square$

Freidlin–Wentzel Theory

Aim: Look at diffusion processes

$$\underset{\mathbb{R}^n}{dx_t^\varepsilon} = \underbrace{b(x_t^\varepsilon)}_{\text{vectorfield}}\, dt + \sqrt{\varepsilon}\, \underbrace{\sigma(x_t^\varepsilon)}_{\mathbb{R}^{n \times m} \text{ mtx}}\, \underset{m\text{-dim BM}}{dW_t}$$

What happens if $\varepsilon \downarrow 0$?

Clearly: $x_t^\varepsilon \longrightarrow x_t^0$ (locally uniformly a.s.)

x_t solution to $\dot{x}_t = b(x_t)$.

Warmup: Additive noise case $\sigma(x_t) = \sigma$ = constant

Thm (Contraction principle)

(E, d) separable, complete. μ_n satisfy LDP with rate I. $T: E \to \tilde{E}$ continuous !!!

Then $T_\# \mu_n$ satisfy a LDP on $\tilde{E}$ with rate fct.

$$\tilde{I}(x) = \inf_{y: T(y) = x} I(y)$$

proof: (1) $\tilde{I}(x)$ has compact sublevelsets.

$$\{x : \tilde{I}(x) \le L\} = \underbrace{T}_{\text{cont.}} \underbrace{\{y : I(y) \le L\}}_{\text{compact}} \quad \text{compact.}$$

② Lower bound in open sets: $\mathcal{O} \subset \tilde{E}$ open

$$\underline{\lim} \tfrac{1}{n} \log T_{\#}\mu_n(\mathcal{O}) = \underline{\lim} \tfrac{1}{n} \log \mu_n(\underbrace{T^{-1}(\mathcal{O})}_{\text{open in } E})$$

$$\geq -\inf_{y \in T^{-1}(\mathcal{O})} I(y)$$

$$= -\inf_{x \in \mathcal{O}} \tilde{I}(x).$$

③ Upper bound for closed. $\mathcal{C} \subset \tilde{E}$ closed.

$$\overline{\lim} \tfrac{1}{n} \log T_{\#}\mu_n(\mathcal{C}) = \overline{\lim} \tfrac{1}{n} \log \mu_n(\underbrace{T^{-1}(\mathcal{C})}_{\text{closed}})$$

$$\underset{\text{upper bd}}{\overset{\text{LD}}{\leq}} -\inf_{y \in T^{-1}(\mathcal{C})} I(y)$$

$$= -\inf_{\mathcal{C}} \tilde{I}. \qquad \square$$

Corollary (Freidlin Wentzell bounds - additive noise)

Suppose x_t^ε solves, b uniformly Lipschitz

$$dx_t^\varepsilon = b(x_t^\varepsilon)\,dt + \sqrt{\varepsilon}\,\sigma(x_t^\varepsilon)\,dw_t.$$

Then the distributions of x_t^ε on $C([0,T])$

satisfy a LDP with rate function

$$I(x) = \inf_{v \in H^1} \int |v_s|^2\,ds.$$

$$\forall t \quad x_t = x_0 + \int_0^t b(x_s)\,ds + \int_0^t \sigma \quad v_s\,ds$$

If σ is invertible. This can be written as

$$I(x) = \int_0^T (\dot{x} - b(x_s))\, a^{-1}\, (\dot{x} - b(x_s))\,ds$$

where $a = \sigma\sigma^T$

proof: want to apply contraction principle to the mapping $T: C_0([0,T]) \to C([0,T])$.

that maps w_s to the solution x_s of

$$x_t = \int_0^t b(x_s)\,ds + x_0 + \quad \sigma w_t$$

Then $x_t^w := T(w)(t)$

$$|x_t^w - x_t^{\bar{w}}| \leq \int |b(x_s^w) - b(x_s^{\bar{w}})|\,ds + |w_t - \bar{w}_t|$$

$$\leq |b|_{Lip} \int_0^t |x_s^w - x_s^{\bar{w}}| + |w_t - \bar{w}_t|$$

$\Rightarrow$ Gronwall gives continuity!

$\Rightarrow$ Contraction principle gives the result! $\square$

Multiplicative noise case

Aim: We want to have a contraction principle that applies to T: that maps w (BM) to solution of

$$dx_t = b(x_t)\,dt + \sigma(x_t)\,dw_t$$

by Approximation.

Comments

- **Ex** proof the LDP in the additive noise case if $b = \nabla B$ and $\sigma = \mathrm{Id}$, using Girsanov Thm & Varadhan Lemma.

- There are many approaches to proof this LDP. E.g.
  - Rough paths
  - Varadhan-Laplace principles. (recent book by Feng & Kurtz)

<u>Thm</u> (Generalised contraction principle)

(E, d) separable metric complete; μ_n satisfy LDP with rate I. $T: E \to \tilde{E}$ measurable. $(\tilde{E}, \tilde{d})$ sep. metric, comp.

Assume that there are continuous mappings $T_m: E \to \tilde{E}$ such that

* (Exponentially good approximation of T)

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}} \frac{1}{n} \log \mu_n\{x \in E : \tilde{d}(T^m x, T(x)) > \delta\} = -\infty$$

* (T^m converges uniformly to T on sublevelsets of I) $\forall L$

$$\sup_{x : I(x) \le L} \{\tilde{d}(T^m(x), T(x))\} \xrightarrow[m\to\infty]{} 0$$

$\Rightarrow T_\# \mu_n$ satisfy a LDP with rate fct.

$$\tilde{I}(x) = \inf_{y : T(y) = x} I(y).$$

proof of the contraction principle:

① Properties of $\tilde{I}$

* restrict T^m, T to $K_L := \{x \in E, I(x) \le L\}$ compact

T^m converges uniformly to T, $\Rightarrow T|_{K_L}$ is cont.

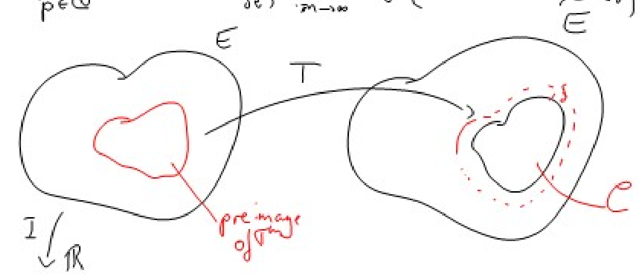
* infimum is attained, i.e. $\tilde{p} \in \tilde{E}$, $\tilde{I}(p) < \infty$

$\Rightarrow \exists\, p \in E : T(p) = \tilde{p}$, $I(p) = \tilde{I}(\tilde{p})$

$\Rightarrow \{\tilde{p} \in \tilde{E} : \tilde{I}(\tilde{p}) \le L\} = \underbrace{T}_{\text{cont. on } K_L}(\underbrace{K_L}_{\text{compact}}) \Rightarrow$ compact.

② $\mathcal{C} \subset \tilde{E}$ closed. Then

$$\inf_{\tilde{p} \in \mathcal{C}} \tilde{I}(\tilde{p}) = \lim_{\delta \downarrow 0} \underline{\lim_{m\to\infty}} \inf \{ I(p) : \tilde{d}(T^m_p, \mathcal{C}) \le \delta \}$$



$\boxed{\geq}$ Pick $\tilde{p} \in \mathcal{C}$ with finite $\tilde{I}$

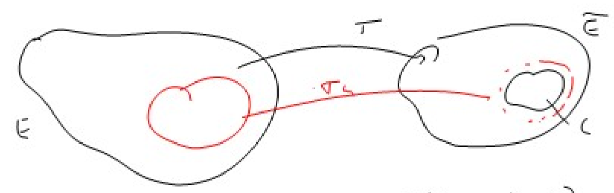
$\exists\, p \in E$ s.th. $T(p) = \tilde{p}$, $I(p) = \tilde{I}(\tilde{p})$

we know that $T_m(p) \to T(p) = \tilde{p}$

$\Rightarrow$ for every δ, there exist m_δ such that $m \ge m_\delta$ $\tilde{d}(T_m(p), \mathcal{C}) \le \delta$.

$\Rightarrow \quad \liminf (I(p), d \ldots) \le I(p) = \tilde{I}(\tilde{p})$

$\boxed{\leq}$



$$\ell := \lim_{\delta \downarrow 0} \varliminf_{m\to\infty} \inf \{ I(p) ; \tilde{d}(p, \mathcal{C}) \leq \delta \}$$

We can find q_m : $I(q_m) \leq \ell + 1$
$I(q_m) \downarrow \ell$.
$\tilde{d}(T_{n_m}(q_m, \mathcal{C})) \leq \frac{1}{m}$.

$K_{\ell+1}$ compact, $\Rightarrow$ $q_m \longrightarrow q$ (after passing to a subsequence)

$$I(q) = \ell \quad \& \; q \in \mathcal{C}.$$

③ Large deviation lower bound.

$\tilde{p} \in \tilde{E}$ bound from below the probability of $\mathcal{B}_{2\delta}(\tilde{p})$

$$\mu_n\left(Tp \in \mathcal{B}_{2\delta}(\tilde{p})\right) \geq \mu_n\left(T^m_p \in \mathcal{B}_\delta(\tilde{p}) \;\&\; \tilde{d}(T^m_p, T_p) \leq \delta\right)$$

$$\geq \underbrace{\mu_n\left(T^m p \in \mathcal{B}_\delta(\tilde{p})\right)} - \underbrace{\mu_n\left(\tilde{d}(T^m_p, T_p) > \delta\right)}$$

by choosing m large enough → make as small as we want.

$\exists\, p$, $Tp = \tilde{p}$, $\tilde{I}(\tilde{p}) = p$

$$\{ \hat{p} : T^m_{\hat{p}} \in \mathcal{B}_\delta(\tilde{p}) \} \supseteq \underbrace{B_{r_m}(p)}$$

$$\Rightarrow \varliminf_n \frac{1}{n} \log \mu_n\left(B_{r_m}(p)\right) \overset{LD}{\geq} - I(p) = -\tilde{I}(\tilde{p}).$$

Upper bd: $\mathcal{C} \subset \tilde{E}$ closed.

$$\mu_n(Tp \in \mathcal{C}) \underset{\forall m}{\leq} \mu_n\left(\tilde{d}(T\hat{p}; \mathcal{C}) \leq \delta\right) + \underbrace{\mu_n\left(\tilde{d}(T^m_p, T_p) > \delta\right)}$$

this can be made arbitrarily small in exponential scale.

$\forall \delta, m$. $\varlimsup_n \frac{1}{n} \log \mu_n\left(\tilde{d}(T^m_p, \mathcal{C}) \leq \delta\right)$

$$\leq - \inf \{ I(p) ; d(T^m_p, \mathcal{C}) \leq \delta \}$$

Then by letting $m \to \infty$ and $\delta \downarrow 0$ and ① we have the result!

Back to SDE:

$$x^\varepsilon_t = x_0 + \int_0^t b(x^\varepsilon_s)\, ds + \int_0^t \sqrt{\varepsilon}\, \sigma(x^\varepsilon_s)\, dw_s.$$

wlog $x_0 = 0$, wlog final time $= 1$! $\quad t_i := \frac{i}{m}$

$T^m : C_0(0,1) \longrightarrow C_0(0,1)$

$T^m w = x \qquad x_t = \sum_{i=0}^{m-1} b(x_{t_i})(t_{i+1}\wedge t - t_i \wedge t) + \sigma(x_{t_i})(w_{t_{i+1}\wedge t} - w_{t_i \wedge t})$

$$= \int_0^t b(x_{s^m})\, ds + \int_0^t \sigma(x_{s^m})\, dw_s .$$

for $t \in [0,1]$; $\frac{\lfloor tm \rfloor}{m}$

Need to check: (1) T_m are continuous.

(2) Uniform Convergence on sublevel set of I.

$$\sup_{w : \|w\|_{H^1} \le \alpha} (T^m w, Tw) \to 0 \quad m \to \infty .$$

(3) Exponentially good approximation

$$\lim_{m\to\infty} \overline{\lim_{\varepsilon \to 0}} \varepsilon \log \mathbb{P}\left(\sup_{0 \le t \le 1} |x_t^{m,\varepsilon} - x_t^\varepsilon| \ge \delta \right) = -\infty .$$

(1) OK ✓

(2) bound on $|x_{s^m} - x_s| \le \int_{s^m}^{s} |b(x_{s^m})| ds$

$$+ \int_{s^m}^{s} |\sigma(x_{s^m})\, \dot{w}| ds$$

Cauchy Schwarz $\le \frac{1}{m} \cdot |b|_{L^\infty} + \sqrt{\frac{1}{m}} |\sigma|_{L^\infty} \cdot \underbrace{|\dot{w}|_{L^2}}_{\le \alpha}$

$$\le \frac{1}{\sqrt{m}} \cdot C$$

$$x_t - x_t^m = \int b(x_t) - b(x_t^m)\, ds$$

↳ Gronwall

⇒ next time!