Projective Resolutions for Smooth G-modules

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Outline

For a locally compact totally disconnected group G which acts on a simplicial complex $\mathcal{X}$ we investigate the following:

- the projective dimension of the category of smooth representations of G,
- explicit projective resolutions for each smooth G-module V,
- explicit finitely generated projective resolutions for each smooth G-module V.

The representations

A representation (π, V) of G over $\mathbb{F}$ is called *smooth* if for all $v \in V$ there exists a compact open subgroup K_v of G such that $\pi(k)v = v$ for all $k \in K_v$.

Denote by $\mathcal{M}(G)$ the category of all smooth representations of G. It is abelian and has enough projectives.

First bound on $\operatorname{proj.dim} \mathcal{M}(G)$

Theorem (Rumynin, H.)

Let G be a locally compact totally disconnected group. Suppose G acts on a simplicial complex $\mathcal{X} = (\mathcal{X}_n)$ such that its geometric realisation $|\mathcal{X}|$ is contractible of dimension n. Suppose further that the stabiliser G_x of any $x \in \mathcal{X}_k$ is open and compact. Then

$$\operatorname{proj.dim}(\mathcal{M}(G)) \leqslant n.$$

Example

1. $\mathrm{SL}_n(\mathbb{K})$, where $\mathbb{K}$ is a non-archimedean local field, with the action on its Bruhat-Tits building $\mathcal{BT}$.
2. Kac-Moody groups with the action on the Davis realisation $\mathcal{D}$ of their buildings.

What do we lack?

For $V \in \mathcal{M}(G)$ the resolution looks like

$$0 \to X_n \otimes V \xrightarrow{d_n} X_{n-1} \otimes V \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} X_0 \otimes V \to V \to 0, \qquad (1)$$

where the X_i's are the $\mathbb{F}$-vector spaces formally spanned by the elements of $\mathcal{X}_{(i)}$, the set of non-degenerate simplices in $\mathcal{X}_i$.

Problem: Even if V is f.g., the modules in (1) do not have to be f.g.

In search of finiteness - the idea of Schneider and Stuhler

Fix a locally compact totally disconnected group G which acts on a simplicial complex $\mathcal{X} = (\mathcal{X}_n)$. New objects are needed:

- an *equivariant cosheaf on* $\mathcal{X}$ is a cosheaf $\mathcal{C} = (\mathcal{C}_x)$ of vector spaces on $\mathcal{X}$ with additional data: a linear map $\mathbf{g}_x : \mathcal{C}_x \to \mathcal{C}_{\mathbf{g}x}$ for all $\mathbf{g} \in G$ and any simplex x, such that:
  - (i) $\mathbf{g}_{\mathbf{h}x} \circ \mathbf{h}_x = (\mathbf{gh})_x$ for any $\mathbf{g}, \mathbf{h} \in G$ and a simplex x.
  - (ii) $\mathcal{C}_x$ is a smooth representation of the simplex stabiliser G_x for any simplex x.
  - (iii) The square
$$\begin{array}{ccc} \mathcal{C}_x & \xrightarrow[\mathbf{g}_x]{} & \mathcal{C}_{\mathbf{g}x} \\ \big\downarrow{\scriptstyle \mathcal{C}(f,x)} & & \big\downarrow{\scriptstyle \mathcal{C}(\mathbf{g}f,\mathbf{g}x)} \\ \mathcal{C}_{\mathcal{X}(f)x} & \xrightarrow{\mathbf{g}_{\mathcal{X}(f)x}} & \mathcal{C}_{\mathcal{X}(\mathbf{g}f)\mathbf{g}x} \end{array}$$
is commutative for all $\mathbf{g} \in G$, simplices $x \in \mathcal{X}_n$ and nondecreasing maps $f : [m] \to [n]$.

Exquisite systems of subgroups

A *system of subgroups* $\mathcal{G}$ of G acting on $\mathcal{X}$ is a datum assigning a subgroup $\mathcal{G}_x$ of the simplex stabiliser G_x to each simplex $x \in \mathcal{X}_n$ such that $\mathbf{g}\mathcal{G}_x\mathbf{g}^{-1} = \mathcal{G}_{\mathbf{g}x}$ for all $\mathbf{g} \in G$ and $x \in \mathcal{X}_n$.

We call the system

- *open* if each $\mathcal{G}_x$ is open in G_x,
- *compact* if each $\mathcal{G}_x$ is compact,
- *contravariant* if for simplices x and y, such that $x \subseteq y$ we have $\mathcal{G}_x \subseteq \mathcal{G}_y$.

Suppose we are given a compact open subgroup $\mathcal{G}_x$ for each vertex $x \in \mathcal{X}_0$ such that

(1) $\mathcal{G}_{\mathbf{g}x} = \mathbf{g}\mathcal{G}_x\mathbf{g}^{-1}$ for all $\mathbf{g} \in G$, $x \in \mathcal{X}_0$ and

(2) $\mathcal{G}_x\mathcal{G}_y = \mathcal{G}_y\mathcal{G}_x$ if x and y are adjacent.

We can extend this to a compact open contravariant system of subgroups by defining:

$$\mathcal{G}_x := \mathcal{G}_{\mathcal{X}(f_0^n)x}\mathcal{G}_{\mathcal{X}(f_1^n)x} \cdots \mathcal{G}_{\mathcal{X}(f_n^n)x} \quad \text{for all} \quad x \in \mathcal{X}_n.$$

A system obtained by this construction is called *an exquisite system*.

The Schneider-Stuhler Resolution

Given a contravariant system of subgroups $\mathcal{G}$ on G and $V \in \mathcal{M}(G)$ we define an equivariant cosheaf $\underset{\sim}{V}^{\mathcal{G}}{}_x = V^{\mathcal{G}_x}$ on $\mathcal{X} = (\mathcal{X}_n)$ by

$$V^{\mathcal{G}_x} \coloneqq \{v \in V \mid \mathbf{g} \cdot v = v \text{ for all } \mathbf{g} \in \mathcal{G}_x\}.$$

We call a finitely generated projective resolution of the form $C_\bullet(\mathcal{X}, \underset{\sim}{V}^{\mathcal{G}})$ *a Schneider-Stuhler resolution*.

Constructing the resolution for n=1

Theorem (Rumynin, H.)

Let G be a locally compact totally disconnected group which acts smoothly on a tree $\mathcal{T}$ such that $\mathcal{T}_{(k)}$, for $k = 0, 1$, has finitely many G-orbits. Then for an admissible $V \in \mathcal{M}(G)$ which is generated by invariants $V^{\mathcal{G}_x}$ for some $x \in \mathcal{T}_0$ and $\mathcal{G}$ a geodesic exquisite system of subgroups, the following complex

$$0 \to C_1(\mathcal{T}, \underset{\approx}{V}^{\mathcal{G}}) \xrightarrow{d_1} C_0(\mathcal{T}, \underset{\approx}{V}^{\mathcal{G}}) \xrightarrow{w} V$$

is a Schneider-Stuhler resolution.

Conjecture - the general case

Conjecture (Rumynin, H.)

Let G be a locally compact totally disconnected group which acts smoothly on a simplicial set $\mathcal{X}_\bullet$ of dimension n. Suppose that a face of a non-degenerate simplex in $\mathcal{X}_\bullet$ is non-degenerate and $|\mathcal{X}|$ admits a CAT(0)-metric such that the faces are geodesic. Then for $V \in \mathcal{M}(G)$ with same assumptions as in Theorem 1 and $\mathcal{G}$ a geodesic exquisite system of subgroups, the following complex

$$0 \to C_n(\mathcal{X}_\bullet, \underset{\sim}{V}^{\mathcal{G}}) \xrightarrow{d_n} C_{n-1}(\mathcal{X}_\bullet, \underset{\sim}{V}^{\mathcal{G}}) \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} C_0(\mathcal{X}_\bullet, \underset{\sim}{V}^{\mathcal{G}}) \xrightarrow{w} V$$

is a Schneider-Stuhler resolution.