Bruhat-Tits Buildings

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The set up

Let (W, S) be a Coxeter system, where S is a set indexed by $\mathcal{I} = \{1, .., n\}$. So W is of the form

$$W = \langle s_1, ..., s_n | (s_i)^2 = 1, (s_i s_j)^{m_{i,j}} = 1 \rangle \text{ for } m_{i,j} \geq 2 \text{ when } i \neq j.$$

For $S' \subseteq S$ we can define a group $W' = \langle S' \rangle$. Then (W', S') is in fact a Coxeter system and W' is called a *special subgroup* of W.

The Coxeter complex $\Sigma(W, S)$

To every Coxeter system we can associate a simplicial complex $\Sigma(W, S)$ defined as follows:

Let $\mathcal{P}$ be the set of all special cosets in W. Then the pair $(\mathcal{P}, \leq)$ is a partially ordered set with $\leq$ being the opposite of the inclusion relation. Simplices in $\Sigma(W, S)$ are given by chains in this poset.

Some examples

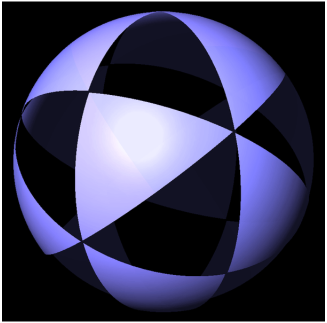


Figure: The Coxeter complex for the group A_3

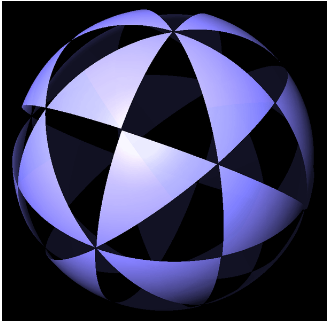


Figure: The Coxeter complex for the group B_3

Chamber complex

A *chamber complex* is a simplicial complex such that all maximal simplices have the same dimension and every pair of such simplices can be connected by a gallery.
The maximal dimension simplices are called *chambers*.
Two chambers C and C' are called *adjacent* if they intersect in a codimension 1 face.

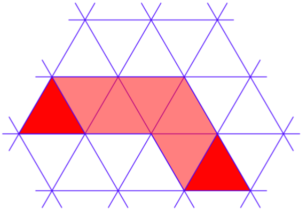


Figure: $\tilde{A}_2$

A Coxeter complex is a chamber complex

Every Coxeter complex is a chamber complex:

- All maximal simplices have the same dimension - they are the singletons $\{w\}, w \in W$. These are the *chambers*. The singleton $\{1\}$ is called the *fundamental chamber*.
- Every two chambers can be connected by a gallery.

Labelling chamber complexes

Chamber complexes can be labelled by a set.

In particular, for a chamber complex $\mathcal{C}$ and a set $\mathcal{I}$, there exists a map $\lambda : \mathcal{I} \to \mathcal{C}$ such that λ assigns an element of $\mathcal{I}$ to each vertex of a chamber C.

What is a building?

Defintion

A *building* Δ is a simplicial complex, which can be expressed as the union of subcomplexes Σ called *aparments* such that the following conditions are satisfied:

(B0) Each apartment is a Coxeter complex.

(B1) For every two simplices $\sigma, \tau \in \Delta$ there exists an apartment Σ containing both of them.

(B2) If Σ and Σ' are apartments both containing τ and σ, then there exists and isomorphism $\phi : \Sigma \to \Sigma'$ which fixes τ and σ pointwise.

If $\mathcal{A}$ is a collection of subcomplexes satisfying the conditions above, then $\mathcal{A}$ is called *a system of apartments*.

A union of a system of apartments is again a system of apartments, therefore for every building Δ we can choose a maximal system of apartments, which is called *the complete system of apartments*.

First consequences of the definition

- Every two apartments are isomorphic.
- Δ is a chamber complex.
- Δ is labellable. Moreover, the isomorphisms in the axiom (B2) can be chosen as label-preserving.

More consequences

There exists a Coxeter matrix M associated to Δ.
Choose a labelling of Δ by a set $\mathcal{I}$ we can write a Coxeter matrix $M_\Sigma = (m_{i,j})$, $i, j \in \mathcal{I}$ for every apartment Σ in the following way:

$$m_{i,j} = \mathrm{diam}(\mathrm{lk}_\Sigma A),$$

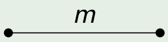
where A is any simplex in Σ of type $\mathcal{I} - \{i, j\}$. The result follows as all apartments are isomorphic.

Reconstructing buildings form Coxeter diagrams

Having a Coxeter matrix associated to Δ, we can talk about a Coxeter diagram associated to Δ.

Example

Take the Coxeter diagram



- For $m = 2$, every apartment is a quadrilateral.
- For $m = 3$, every apartment is a hexagon.
- For $m = \infty$, every apartment is a line and Δ is a tree with no endpoints. In fact, there is a result which states that every building of this type is a tree.

The building of $SL_2(\mathbb{Q}_p)$

- it is a $p+1$-regular tree.
- vertices are equivalence classes of $\mathcal{O}$-lattices $[\Lambda]$ in $\mathbb{Q}_p^2$.
- there is an edge between 2 vertices $[\Lambda]$ and $[\Lambda']$ if $\pi\Lambda' \subsetneq \Lambda \subsetneq \Lambda'$.

From buildings to BN-pairs

Let us fix some notation:

- Δ - a thick building.
- (Σ, C) - respectively the fundamental apartment and the fundamental chamber of Δ.
- G - a group which acts strongly transitively on Δ by type-preserving automorphisms.
- W - a group acting on Σ by type-preserving automorphisms.
- S a set of reflections through the codim 1 faces of C. Note: (W, S) is a Coxeter system and $\Sigma \cong \Sigma(W, S)$.
- λ - canonical labelling of $\Sigma(W, S)$ with S as the set of labels.

- Also take the following subgroups of G:
  - $B = \{g \in G | gC = C\}$.
  - $N = \{g \in G | g\Sigma = \Sigma \text{ setwise}\}$.
  - $T = \{g \in G | g\Sigma = \Sigma \text{ pointwise}\}$.

We immediately spot the following properties of B, N and T:

- $T \lhd N$.
- $W \cong N/T$.
- $T = B \cap N$.

Reconstructing Δ from G, B, N and S

Let $\text{ch}(\Delta)$ denote the set of chambers of Δ. Then we see that we have a set isomorphism:

$$\text{ch}(\Delta) \cong G/B$$

$$\text{by } gC \mapsto gB$$

How does the adjacency of chambers transfer on the right?

Lemma

Two chambers gB and $g'B$ are s-adjacent if and only if $g^{-1}g' \in P_s$, where P_s is the stabiliser of the face of gC of type $S - \{s\}$.

Voila: we have the chamber complex of Δ described only by a group-theoretic property!

Stabilisers of faces of C

The stabiliser P_s of a face of C of type $S - \{s\}$ can be written explicitly in terms of B:

$$P_s = B \cup BsB = B < s > B$$

.

In fact this can be generalised:

Lemma

The stabiliser $P_{S'}$ of a face of C of type $S - S'$ has the form:

$$P_{S'} = B < S' > B = BW'B$$

.

There is a poset isomorphism between the set of subsets of S and the partially ordered set of subgroups of the form P'_S given by the obvious map.

In particular, this implies that Δ, as a poset, is isomorphic to the set of cosets of the P'_S's.

Last steps towards the BN-pair

Need:

(BN1) $BwB \cdot BsB \subseteq BwB \cup BwsB$ for any $w \in W$ and $s \in S$.

(BN2) $s^{-1}Bs \not\subseteq B$ for any $s \in S$.

Both axioms hold! In fact, (BN2) is saying in a group theoretic language that Δ is thick.