Representations of complete Kac-Moody groups

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Kac-Moody Groups

Let A be a generalised Cartan matrix. To any field $\mathbb{K}$ and a root datum $\mathcal{D}$ of type A one can combinatorially associate a Kac-Moody group $G_{\mathcal{D}}(\mathbb{K})$. A few topologies can be put on $G_{\mathcal{D}}(\mathbb{K})$. Taking completions, we obtain different types of **complete Kac-Moody groups**:

- Carbone-Garland completion G^{cg} with respect to the weight topology,
- Caprace-Rémy completion G^{cr} with respect to the building topology,
- Ronan-Rémy completion G^{rr},
- Capdeboscq-Rumynin completion G^{car} by putting a local pro-p topology,

which are connected by a sequence of continuous open homomorphisms:

$$G^{car} \twoheadrightarrow G^{cr} \twoheadrightarrow G^{cg} \twoheadrightarrow G^{rr}.$$

Kac-Moody groups have (B,N)-pairs and Bruhat-Tits buildings. There are also **topological groups of Kac-Moody type** - a more general class of groups with a generalised (B,N)-pair structure, which resemble complete Kac-Moody groups [1].

Smooth Representations

Throughout G is a locally compact totally disconnected topological group and $\mathbb{F}$ is a field of characteristic zero.
We call a pair (π, V) a **smooth representation** of G if:

(1) V is a vector space over $\mathbb{F}$ and $\pi : G \to \mathrm{Aut}_{\mathbb{F}}(V)$ is a homomorphism,

(2) For every $v \in V$, there exists a compact open subgroup $K_v \leq G$, such that $\pi(k)v = v$, for all $k \in K_v$.

Smooth representations of a locally compact totally disconnected group form a category - $\mathcal{M}(G)$. It has some particularly nice properties:

- $\mathcal{M}(G)$ is abelian.
- $\mathcal{M}(G)$ has enough projectives.
- $\mathcal{M}(G)$ is Noetherian.

In particular, for each object in $\mathcal{M}(G)$ we can construct a projective resolution.

Projective Dimension

Theorem 1. *[1] Let G be a locally compact totally disconnected group acting on an n-dimensional simplicial complex (set) $\mathcal{X}_\bullet$ with contractible geometric realisation $|\mathcal{X}|$. Suppose the stabilisers of non-degenerate simplices are compact and open in G. Then*

$$\mathrm{proj.\,dim}(\mathcal{M}(G)) \leq n.$$

Let $(\pi, V) \in \mathcal{O}b(\mathcal{M}(G))$. The explicit projective resolution of V looks like:

$$X_n \otimes V \to X_{n-1} \otimes V \to \cdots \to X_0 \otimes V \to V \to 0$$

with

$$X_k \xleftarrow{\cong} \sum_{x \in \mathcal{X}_{(k)}(G)} \mathbb{F}G \underset{\mathbb{F}G_x}{\otimes} \mathbb{F}[x], \quad \alpha[g \cdot x] \leftarrow\!\shortmid g \otimes \alpha[x],$$

where X_k is the $\mathbb{F}$-space spanned by non-degenerate k-simplices, $\mathbb{F}[x]$ is the space spanned by the non-degenerate k-simplex x and $G_x \leq G$ is the stabiliser of x.

Corollary 2. *[1] If G is a complete Kac-Moody group or a topological group of Kac-Moody type, then*

$$\mathrm{proj.\,dim}\,\mathcal{M}(G) \leq \sup_{J \in \mathcal{S}ph(S)} |J|.$$

Equivariant Cosheaves

For a locally compact totally disconnected group G acting on a simplicial set $\mathcal{X}_\bullet$ we can form another interesting category - $\mathrm{Csh}_G(\mathcal{X}_\bullet)$, the category of **G-equivariant cosheaves on $\mathcal{X}_\bullet$**.

Definition 3. *A **G-equivariant cosheaf** is a cosheaf $\mathcal{C}$ with an additional data: a linear map $g_x = g(\mathcal{C})_x : \mathcal{C}_x \to \mathcal{C}_{gx}$, for any $g \in G$ and any simplex x. This data satisfies three axioms:*

(1) $g_{hx} \circ h_x = (gh)_x$ for any $g, h \in G$ and a simplex x.
(2) $\mathcal{C}_x$ is a smooth representation of G_x for any simplex x.
(3) The square

$$\begin{array}{ccc} \mathcal{C}_x & \xrightarrow{g_x} & \mathcal{C}_{gx} \\ \downarrow{\scriptstyle \mathcal{C}(f,x)} & & \downarrow{\scriptstyle \mathcal{C}(gf,gx)} \\ \mathcal{C}_{\mathcal{X}(f)x} & \xrightarrow{g_{\mathcal{X}(f)x}} & \mathcal{C}_{\mathcal{X}(gf)gx} \end{array}$$

commutes, for all $g \in G$, simplices $x \in \mathcal{X}_n$ and nondecreasing maps $f : [m] \to [n]$.

Localization

We have two functors:

$$\mathcal{L} : \mathcal{M}(G) \to \mathrm{Csh}_G(\mathcal{X}_\bullet), \ (\pi, V) \mapsto \underset{\sim}{V}$$

and

$$\mathcal{H} : \mathrm{Csh}_G(\mathcal{X}_\bullet) \to \mathcal{M}(G), \ \mathcal{C} \mapsto H_0(\mathcal{X}_\bullet, \mathcal{C}).$$

Theorem 4. *[1] Let Σ be a class of morphisms f in $\mathrm{Csh}_G(\mathcal{X}_\bullet)$, such that $\mathcal{H}(f)$ is an isomorphism. If $|\mathcal{X}|$ is connected, then*

$$\mathcal{H}[\Sigma^{-1}] : \mathrm{Csh}_G(\mathcal{X}_\bullet)[\Sigma^{-1}] \to \mathcal{M}(G)$$

is an equivalence of categories, where $\mathcal{H}[\Sigma^{-1}]$ is the functor induced from $\mathcal{H}$ on the category of left fractions $\mathrm{Csh}_G(\mathcal{X}_\bullet)[\Sigma^{-1}]$.

Conjecture 5. *There is a quotient of categories*

$$H_* : \mathcal{SM}(G) \to D^{co}(\mathcal{M}(G)),$$

where $D^{co}(\ \ )$ is the coderived category and $\mathcal{SM}(G)$ is the category of simplicial representations of G.

Homological Duality

The space of all locally constant, compactly supported functions $f : G \to \mathbb{F}$ with respect to the convolution product is an $\mathbb{F}$-algebra $\mathscr{H}_G$, called **the Hecke algebra of G** . An $\mathscr{H}_G$-module $(M, \cdot)$ is called **smooth** if $\mathscr{H}_G \cdot M = M$. Denote the category of smooth $\mathscr{H}_G$-modules by $\mathcal{M}(\mathscr{H})$. There is an **equivalence** of categories:

$$\mathcal{M}(G) \cong \mathcal{M}(\mathscr{H}_G).$$

Theorem 6. *[1] $\mathscr{H}_G$ is a dualising bimodule.*

Conjecture 7. *[1] Let G be a topological group of Kac-Moody type with B compact open. For a simple module $M \in \mathcal{M}(G)^B$, its dual $M^\vee$ is a simple module in $\mathcal{M}(G)^B$. Furthermore, when viewed as $\mathscr{H}_{B\backslash G/B}$-bimodules, M and $M^\vee$ are twists of each other under the Iwahori-Matsumoto involution, where $\mathcal{M}(G)^B$ is the full subcategory of $\mathcal{M}(G)$ consisting of smooth representations generated by their B-invariants and $\mathscr{H}_{B\backslash G/B}$ is the subalgebra of $\mathscr{H}_G$ consisting of B-biinvariant functions.*

References

[1] K. Hristova, D. Rumynin, Kac-Moody groups and Cosheaves on Davis building, preprint, arXiv:1704.07880.