Groups, graphs and magic

Katerina Hristova

University of Warwick

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Women in Maths

- Hypatia
- Maria Agnesi
- Sophie Germain
- Mary Somerville
- Ada Lovelace
- Sofia Kovalevskaya
- Emmy Noether

many, many more..

Augusta Ada Byron

- Born on 10 December 1815 in London, daughter of Lord Byron
- Got her education in science and mathematics by private tutors amongst which was Mary Somerville
- Married William King on 8 July 1835
- Her husband received the title Earl of Lovelace. Thus, she became Countess of Lovelace
- Died on 27 November 1852 at the age 36

Ada's Achievements

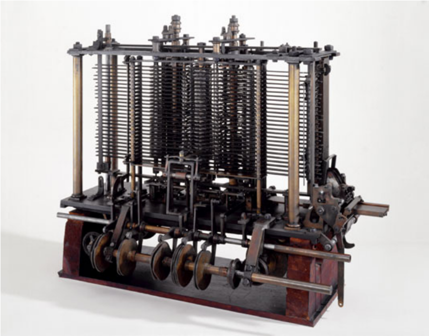


Figure: Reproduction of "The Analytical Engine" designed by Babbage

- Worked together with Charles Babbage, who called her "enchantress of numbers"
- Developed what is believed to be the first computer algorithm the idea of which was to compute Bernoulli numbers
- Realised that the engine can not only be used as a computational machine, but also it can be programmed to do other things, i.e. produce music

Emmy Noether

- Born on 23 March 1882 in Erlagen, Germany
- Got her education in mathematics at the University of Erlagen
- Taught there for 7 years, moved to University of Gottingen and then to Bryn Mawr College, Pennsylvania
- Died on 14 April 1935 at the age 53

The significance of Emmy Noether's work

- Called "*the most significant creative mathematical genius thus far produced since the higher education of women*" by Albert Einstein
- Worked on algebraic invariant theory, Galois theory (in particular on the Inverse Galois problem where she made significant progress)
- Gave the first definition of a commutative ring in her book "Theory of Ideals in Ring Domains" (1921)
- Introduced the concepts of ideals in commutative rings and analysed the ascending chain condition on ideals which led to the discovery of the so called Noetherian rings
- Worked on representation theory, key discoveries in the representation of an algebra. Also worked noncommutative algebra, topology, physics etc...

Group actions on graphs

Let G be a group and Γ an oriented graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$.

Defintion

G acts on Γ if G acts on $V(\Gamma)$ and $E(\Gamma)$. In other words, an action of a group on a graph is a homomorphism $\phi : G \to \text{Isom}(\Gamma)$.

Example

C_2 acting on a segment by permuting the vertices.

We assume throughout that the actions are with no inversion.

Two important graphs connected to the group action:

- The quotient graph Δ is the graph Γ mod G.
- The fundamental domain of the action is a subgraph X of Γ such that $X \cong \Gamma$ mod G.

An amalgam of two groups

Let G_1, G_2 be groups with presentations $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Suppose A is another group such that $\iota_1 : A \to G_1$ and $\iota_2 : A \to G_2$ are monomorphisms.

Defintion

The *amalgamated product* of G_1 and G_2 over A is the group defined by the following presentation:

$$\langle S_1, S_2 \mid R_1, R_2, \iota_1(a) = \iota_2(a), \text{ for all } a \in A \rangle$$

denoted $G_1 *_A G_2$.

Example

- free products are amalgams over the trivial subgroup
- van Kampen's theorem

Main Theorem

There exists a one to one correspondence between the following sets:

$$\left\{\begin{array}{l}\text{Groups acting on trees with}\\ \text{fundamental domain a segment}\end{array}\right\} \longleftrightarrow \left\{\text{Groups of the form } G_1 *_A G_2\right\}$$

Examples: The infinite dihedral group D_∞

The infinite dihedral group is defined in the following way:

$$D_\infty = \langle a, b | a^2 = 1, aba = b^{-1} \rangle.$$

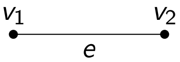
A simple map sending $a \mapsto x$ and $ab \mapsto y$ enables us to rewrite the presentation above as:

$$D_\infty = \langle x, y | x^2 = y^2 = 1 \rangle.$$

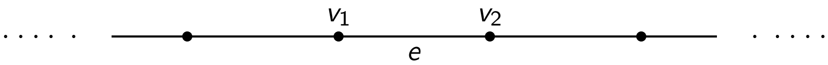
In particular, we have 2 generators with no relations between them and each of them generates a group of order two. What this means is that:

$$D_\infty \cong \mathbb{Z}/2 *_1 \mathbb{Z}/2 = \mathbb{Z}/2 * \mathbb{Z}/2.$$

There exists a tree Γ on which D_∞ acts with fundamental domain a segment. Let us find Γ.



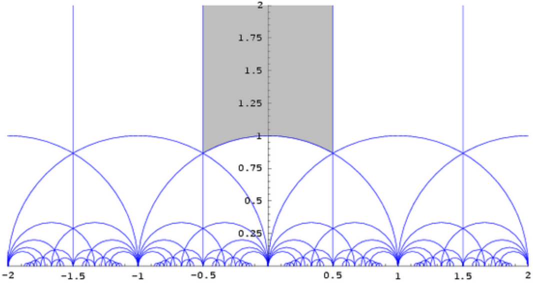
We know that $\text{Stab}(v_1) = \mathbb{Z}/2$, $\text{Stab}(e) = \{1\}$ $\text{Stab}(v_2) = \mathbb{Z}/2$. Thus Γ is the bi-infinite line:



Examples: The modular group $SL_2(\mathbb{Z})$

$SL_2(\mathbb{Z})$ acts on the upper-half plane $\mathbb{H}$ on the left by Mobius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}, \quad z \in \mathbb{H}.$$



Take the segment $\gamma \in \mathbb{H}$ with endpoints $v_1 = e^{\pi/3i}$ and $v_2 = i$. Using the action above we find:

- $\mathrm{Stab}(v_1) = \mathbb{Z}/6$
- $\mathrm{Stab}(e) = \mathbb{Z}/2$
- $\mathrm{Stab}(v_2) = \mathbb{Z}/4$

Thus, $\mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4$.

Generalisations

This can be generalised to taking trees instead of segments, and amalgamated products of n groups. The theory behind this beautiful result is called Bass-Serre theory.

Thank you for your attention!

The End