Errata and additional material for
Infinite-Dimensional Dynamical Systems

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Errata in the book

p 25, l 13 dx is missing from two integrals

p 43, equation (2.3) should read $T(0) = \mathrm{id}$.

p 45, l 9 delete "a" before "locally"

p 46, l5- change $x(t)$ to $x(s)$

p 52, l 13 RHS should be 0

p 75, l 7- closed ball required (should be $\overline{B}(0,1)$)

p 77, l 8- Should be $\inf q(x)$ and $\sup w(x)$, not vice versa

p 80, l 3- Should be $\bar{x}_n \to x$ and $A\bar{x}_n \to \bar{y}$

p 103, l 9 Should be $\|e_j\|$

p 105, l 5- It is not true that the Alaoglu weak-* compactness theorem is valid in any Banach space, as the following example shows (thanks to Vittorino Pata for this example): take $X = l^\infty$, and consider the sequence of functionals $L_n : X \to \mathbb{R}$ defined by

$$L_n(\mathbf{x}) = x_n \qquad \text{when} \qquad \mathbf{x} = (x_1, x_2, x_3, \ldots).$$

The L_n is clearly a bounded sequence in $(l^\infty)^*$, but does not have a weakly-* convergent subsequence. However, Corollary 4.19 (Reflexive weak compactness) *does* hold as stated, i.e. is true for *any* reflexive (not necessarily separable) Banach space - this is where Yosida (1980) provides a proof.

p 109, equation (5.1) and (5.3) on page 110   all derivatives ($\mathrm{d}/\mathrm{d}x_j$) should be partial derivatives ($\partial/\partial x_j$).

p 112,   Definition 5.2 - should say that the derivatives of ϕ_n should converge uniformly to 'the corresponding derivatives of ϕ'

p 115, l 5   should be Theorem 5.4

p 125, l 1   should be 'a function $u \in H^k(\mathbb{R}^m_+)$'

p 132, l 8-   the integral is between -1 and 1

p 160, l 6-   shoudl be u instead of $u(t)$

p 161, l 4-   (5.14) (not (5.13))

p 176, l 4-   The use of λ in Proposition 6.15 is required since we cannot prove immediately H^2 regularity on the whole of D^+, but only on λD^+ for some $0 < \lambda < 1$.

p 190, l 8-   should be $\mathrm{d}u_h/dt \to \mathrm{d}u/dt$

p 196, l 14   change ψ to $\psi(t)$

p 203ff   Section 7.4.3: in fact it is possible to make estimates on $u_n - u_m$, following the calculations in section 7.4.2, and show directly that $\{u_n\}$ is a Cauchy sequence in $L^2(0,T;V)$ and $L^\infty(0,T;H)$, removing the need to extract subsequences. However, the presentation as it stands closely parallels the treatment in subsequent chapters when the subsequence method is necessary.

p 204, l 12   weak-* convergence of $P_n f$ to f in $L^2(0,T;V^*)$ follows using a much simpler argument. In fact, a straightforward application of Lebesgue's dominated convergence theorem (theorem 1.7 (iii)) shows that $P_n f$ converges strongly to f in $L^2(0,T;V^*)$.

p 205, l 9-   change to 'suppose that x_n is given by the constant sequence' p 207, l 8 should be '$t \in [0,T]$'

p 229, equation (8.30) the expression $[1 + \|u\|_{H^1} + \|v\|_{H^1}]$ should be raised to the power of γ rather than $\frac{1}{2}$: the last line of the inequality in the proof should end $(1 + |u|^{2\gamma}_{L^{2q\gamma}} + |v|_{L^{2q\gamma}})^{2\gamma}$ rather than what is written.

p231, Exercise 8.1 You need to assume that X and Y are continuously embedded in some other Banach space Z. [If $X \cap Y$ is dense in both X and Y then we have true equality $(X \cap Y)^* = X^* + Y^*$; otherwise one should say that elements of $(X \cap Y)^*$ can be formed by adding the restrictions to $X \cap Y$ of elements of X^* and Y^*.]

p 235, l 2- change to 'neglecting all the nonlinear terms in (9.1) and taking f to be time-independent'

p 248 we in fact need to take $w \in C^1(0,T;\mathbb{C}^1(\Omega))$ in the argument showing the convergence of $B(u_n,u_n)$ to $B(u,u)$, and then use the density of $C^1(0,T;\mathbb{C}^1(\Omega))$ in $L^q(0,T;V)$. In fact it is better to take $w = \sum_{j=1}^N c_j(t)w_j$, with $c_j(t)$ continuous: for such w it is easy to show that

$$\int_0^T \langle P_n B(u_n,u_n), w\rangle \, \mathrm{d}t \to \int_0^T \langle B(u,u), w\rangle \, \mathrm{d}t,$$

and then since such w are dense in $L^p(0,T;V)$ we obtain convergence of $P_n B(u_n,u_n)$ to $B(u,u)$ in one step.

p 274, Proposition 10.12 second sentence should end 'then $\omega(u_0)$ is a single equilibrium point'.

p 275, Theorem 10.13. Equation (10.20) should read

$$\mathcal{A} = \mathcal{A} \cap W^s(\mathcal{E}) = \mathcal{A} \cap \bigcup_{z \in \mathcal{E}} W^s(z).$$

p 276, l 1 change 'double equality' to 'the two equalities'

p 276, l 9- change '$u_0 \in \mathcal{A}$' to '$v_0 \in \mathcal{A}$'

p 277   the example is wrong (thanks to Prof. Grzegorz Lukaszewicz of Warsaw University for pointing this out). The calculations given are in fact for

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= -zy \\ \mathrm{d}y/\mathrm{d}t &= zx \\ \mathrm{d}z/\mathrm{d}t &= -\mu z|z|. \end{aligned}$$

However, there are some subtleties here. In fact for this example every point has a compact ω limit set, even though the attractor $z \equiv 0$ is not compact, and this is why Proposition 10.14 and its corollary still apply: that problems can arise otherwise is shown in the simpler system $\dot{z} = -z^2$ with $\dot{x} = zx$ which has solution $x(t) = x_0(1 + z_0 t)$ and $z(t) = z_0/(1 + z_0 t)$: although on $z \equiv 0$ all points are stationary, the x component of every solution has constant speed $x_0 z_0$. An example that does away with the need for such subtleties – since it does in fact have a compact global attractor – is

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= (1 - (x^2 + y^2))x - yz \\ \mathrm{d}y/\mathrm{d}t &= (1 - (x^2 + y^2))y + xz \\ \mathrm{d}z/\mathrm{d}t &= -\mu z|z|, \end{aligned}$$

or in polar coordinates

$$\begin{aligned} \mathrm{d}r/\mathrm{d}t &= r(1 - r^2) \\ \mathrm{d}\theta/\mathrm{d}t &= z \\ \mathrm{d}z/\mathrm{d}t &= -\mu z|z|. \end{aligned}$$

p 287, l 7-   (8.27) (not (8.26)); also in l 1-, should be $2l + 1$ not $l + 1$

p 288, l 2-   * missing above the arrow (it's weak-* convergence)

p 294, equation (11.17)   the central term in the equality should be $\int_\Omega f(u)u_t \, \mathrm{d}x$.

p 354, l 6 of Notes   the reference for fractal dimension is to Clark Robinson's 1995 book "Dynamical Systems".

p 354, l -6   instead of the paper by Blinchevskaya & Ilyashenko (which still has not appeared), see instead: Chepyzhov & Ilyin: A note on the fractal dimension of attractors of dissipative dynamical systems. *Nonlinear Anal.* **44** (2001), no. 6, Ser. A: Theory Methods, 811–819 p 381, equation (14.34) the term in k should be $|k|^{1/2}$.

p 390, Definition 15.2 (Strong Squeezing Property) the strictly positive constant k that occurs in (15.9) depends only on the projection P (i.e. only on n).

p 391, l 4ff the two displayed equations should read

$$|Qu - Qv| \leq \mathrm{e}^{-kt}|Qu_t - Qv_t|$$

and

$$|Qu - Qv| \leq 2R_H\mathrm{e}^{-kt},$$

the conclusion being that $Qu = Qv$. The result then follows as before.

p 391, Definition 15.4 (i), $B(0,\rho) \cap PH$ should be replaced by

$$PH \setminus [B(0,\rho) \cap PH]\,,$$

the idea being that the portion of the 'flat space' PH 'outside' the absorbing ball $B(0,\rho)$ is invariant.

p 394, l2 we need '$t \geq t_0(Y)$'. It would be possible to consider $B(0,|u_0|)$ instead of a general bounded set Y, and thus obtain more clearly

$$\mathrm{dist}(S(t)u_0, \mathcal{M}) \leq C(|u_0|)\mathrm{e}^{-kt}.$$

[A more careful proof of exponential convergence can be used to show that in fact

$$\mathrm{dist}(S(t)u_0, \mathcal{M}) \leq C\,\mathrm{dist}(u_0, \mathcal{M})\mathrm{e}^{-kt},$$

see, for example, Chow et al. (1992).]

p 418, l5 while checking that φ satisfies the uniqueness property, the left-hand side should be $\varphi(t;\varphi(s;x))$.

Solutions to Exercises

Some numbering problems for Chapter 3: Solution 3.8 is for Exercise 3.7; Solution 3.9 for Exercise 3.8; and Solution 3.7 is for Exercise 3.9.

The solution for Exercise 8.1 is incorrect, since the application of the Hahn-Banach Theorem does not give a linear functional on X, since it extends a linear functional on $Z = X \cap Y$ which is continuous with respect to the norm of Z, but not that of X. Thanks to Vittorino Pata for a helpful email correspondence to clear this up.

We need to assume in addition that X and Y are continuously embedded in some other Banach space Z. Now take $f \in (X \cap Y)^*$. Now consider the subspace D of $X \times Y$ consisting of vectors of the form (w, w) with $w \in X \cap Y$.

Define a bounded linear functional h on D by

$$h(w, w) = f(w).$$

Now use the Hahn-Banach theorem to extend h to a linear functional g on $X \times Y$, and set

$$f_1(w) = g(w, 0)$$

and

$$f_2(w) = g(0, w).$$

Then $f_1 \in X^*$ and $f_2 \in Y^*$, and $g = f_1 + f_2$. Then for $w \in X \cap Y$, $f = f_1 + f_2$.

Clearly if $X \cap Y$ is dense in X and Y then f_1 and f_2 are uniquely defined by their restrictions to $X \cap Y$, and so the equality $(X \cap Y)^* = X^* + Y^*$ is meaningful. Otherwise, as noted above, we in fact have $f = f_1|_{X \cap Y} + f_2|_{X \cap Y}$, where $f_1 \in X^*$ and $f_2 \in Y^*$.

Additional material

There is a very elegant formulation of an existence result for global attractors (cf. theorem 10.5) that is due to Crauel (2001) in the case of random attractors.

Theorem 1. *There exists a global attractor $\mathcal{A}$ iff there exists a compact attracting set K, and then $\mathcal{A} = \omega(K)$.*

Note that the condition of a compact attracting set is much weaker than the existence of a compact absorbing set. The proof requires the following simple lemma:

Lemma 2. *If K is a compact set and x_n is a sequence such that*

$$\text{dist}(x_n, K) \to 0$$

then $\{x_n\}$ has a convergent subsequence whose limit lies in K.

As a first step to proving this new theorem first we reprove proposition 10.3 under the weaker condition here.

Proposition 3. *If there exists a compact attracting set K then the ω-limit set $\omega(X)$ of any bounded set X is a non-empty, invariant, closed subset of K. Furthermore $\omega(X)$ attracts X.*

Proof. To see that $\omega(X)$ is non-empty choose some point $x \in X$. Then since K is attracting

$$\text{dist}(S(n)x, K) \to 0.$$

It follows that for some sequence $n_j \to 0$

$$S(n_j)x \to x^* \in K.$$

As the intersection of a decreasing sequence of closed sets $\omega(X)$ is clearly closed. To show that $\omega(X) \subset K$ suppose that $t_n \to \infty$, $x_n \in X$ and

$$S(t_n)x_n \to y.$$

Then since K is attracting

$$\text{dist}(S(t_n)x_n, K) \to 0,$$

implying that a subsequence of $S(t_n)x_n$ converges to a point in K. Since the sequence itself converges it follows that $y \in K$. So $\omega(X)$ is compact.

Now suppose that $\omega(X)$ does not attract X. Then there exists a $\delta > 0$ and a sequence of t_n such that

$$\text{dist}(S(t_n)X, \omega(X)) > \delta,$$

and hence $x_n \in X$ such that

$$\operatorname{dist}(S(t_n)x_n, \omega(X)) > \delta. \tag{1}$$

However, the argument above shows that a subsequence of $\{S(t_n)x_n\}$ converges to some point z. By (1) we should have

$$\operatorname{dist}(z, \omega(X)) \geq \delta,$$

while by definition $z \in \omega(X)$. So $\omega(X)$ attracts X. □

Now observe that

$$A \subseteq B \qquad \Longrightarrow \qquad \omega(A) \subseteq \omega(B), \tag{2}$$

and that since $\omega(X)$ is invariant

$$\omega[\omega(X)] = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S(s)\omega(X)} = \omega(X). \tag{3}$$

Proof. (Proof of theorem 1). It follows from the previous proposition that $\omega(K)$ is non-empty, compact, invariant, and attracts K. So all we have to prove is that $\omega(K)$ attracts X. Since $\omega(X)$ attracts X it suffices to show that $\omega(X) \subset \omega(K)$. But this follows immediately from (2) and (3). □