Solutions to Exercises

Chapter 1

1.1 Let $\{x_j\}$ be a countable dense subset of X, and let $\{y_j\}$ be a countable dense subset of Y. Then the countable collection $\{(x_j, y_k)\}$ is dense in $X \times Y$, since for any $(x, y) \in X \times Y$ and any $\epsilon > 0$ there exist x_j and y_k with

$$\|x - x_j\|_X < \epsilon/2 \qquad \text{and} \qquad \|y - y_k\|_Y < \epsilon/2,$$

and so

$$\|(x_j, y_k) - (x, y)\|_{X \times Y} \leq \epsilon.$$

It follows that $X \times Y$ is separable, and by induction it follows that any finite product of separable spaces is separable.

If M is a linear subspace of X then let $\{x_j\}$ be a countable subset of X such that for each $x \in X$ there is an x_j such that $|x - x_j| < \epsilon$. Now discard any element x_j of this collection for which $B(x_j, \epsilon)$ does not intersect M. For each remaining x_j, it follows that there exists an element $m_j \in M$ such that $B(m_j, 2\epsilon) \supset B(x_j, \epsilon)$. Thus this collection $\{m_j\}$ has the property that for each element $m \in M$ there exists an m_j such that $|m - m_j| < 2\epsilon$. Applying this construction for the sequence $\epsilon_n = 2^{-n}$ gives a countable dense subset of M, as required.

1.2 Cover X with the collection of open balls

$$\bigcup_{x \in X} B(x, \epsilon).$$

Since X is compact it follows that there exists a finite covering by such balls:

$$X \subset \bigcup_{j=1}^{N} B(x_j, \epsilon).$$

It follows that for each $x \in X$ there exists an x_j with $|x - x_j| < \epsilon$ as required.

1.3 We first consider the case of Ω bounded. If $u \in C_c^0(\Omega)$ then clearly $u = 0$ on $\partial\Omega$; it follows that if $u_n \in C_c^0(\Omega)$ converges to u uniformly on Ω then $u = 0$ on $\partial\Omega$ too. We now show that any function in

$$C_0^0(\Omega) = \{u \in C^0(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$$

can be arrived at in this way and hence that this space is the completion of $C_c^0(\Omega)$ in the sup norm. Let θ be the continuous function

$$\theta(x) = \begin{cases} x, & x \geq 1, \\ 2x - 1, & 1 > x > \frac{1}{2}, \\ 0, & x \leq \frac{1}{2}, \end{cases}$$

and define

$$u_\epsilon(x) = \theta(|u(x)|/\epsilon)u(x).$$

Clearly u_ϵ is continuous on Ω, and since u is uniformly continuous on Ω there exists a δ such that

$$\text{dist}(x, \partial\Omega) < \delta \qquad \Rightarrow \qquad |u(x)| < \epsilon/2,$$

that is, such that $u_\epsilon(x) = 0$ when $\text{dist}(x, \partial\Omega) < \delta$. It follows that $u_\epsilon \in C_c^0(\Omega)$, and since,

$$|u(x) - u_\epsilon(x)| \leq \epsilon,$$

u_ϵ converges uniformly to u on Ω.

It follows that $C_0^0(\Omega) \neq C_c^0(\Omega)$ is the completion of $C_c^0(\Omega)$ in the sup norm, and $C_c^0(\Omega)$ is therefore not complete.

When $\Omega = \mathbb{R}^m$ the limit of any convergent sequence of functions in $C_c^0(\mathbb{R}^m)$ must tend to zero as $|x| \to \infty$. This is clear, since given $\epsilon > 0$ there exists an N such that $|u_n - u| \leq \epsilon$ for all $n \geq N$. In particular, u_N is zero for all $x > R_N$, say, and so $|u| \leq \epsilon$ for all $x > R_N$. The space of all such u,

$$C_0^0(\mathbb{R}^m) = \left\{u \in C_b^0(\Omega) : u(x) \to 0 \text{ as } |x| \to \infty\right\},$$

is the appropriate completion of $C_c^0(\mathbb{R}^m)$. For any $u \in C_0^0(\mathbb{R}^m)$, we can use the argument above to find an approximating sequence of $u_\epsilon \in C_c^0(\mathbb{R}^m)$.

1.4 If $\{f_j\}$ is Cauchy in the $\|\cdot\|_{\mathbf{c}}$ norm then it is Cauchy in each $C^n(\overline{\Omega})$ norm. Since each $C^n(\overline{\Omega})$ is complete, $f_j \to f$ in each of these spaces, so that $f \in C^n(\overline{\Omega})$ for every n and thus $f \in C^\infty(\overline{\Omega})$. It remains to show that in fact

$$\|f\|_{\mathbf{c}} < \infty$$

and that

$$\|f_j - f\|_{\mathbf{c}} \to 0$$

as $j \to \infty$. Since $\{f_j\}$ is Cauchy it certainly follows that for $j, k \geq N$ we have

$$\sum_{n=1}^{l} c_n \|f_j - f_k\|_{C^n(\overline{\Omega})} < \epsilon$$

for each $l < \infty$, and taking the limit as $k \to \infty$ gives

$$\sum_{n=1}^{l} c_n \|f_j - f\|_{C^n(\overline{\Omega})} < \epsilon. \tag{S1.1}$$

Using the triangle inequality in each $C^n(\overline{\Omega})$, $0 \leq n \leq l$, shows that

$$\sum_{n=1}^{l} c_n \|f\|_{C^n(\overline{\Omega})} < \epsilon + \sum_{n=1}^{l} c_n \|f_j\|_{C^n(\overline{\Omega})},$$

and so

$$\|f\|_{\mathbf{c}} \leq \epsilon + \|f_j\|_{\mathbf{c}}.$$

Since (S1.1) holds for all l, we can let $l \to \infty$ to show that

$$\sum_{n=1}^{\infty} c_n \|f_j - f\|_{C^n(\overline{\Omega})} < \epsilon,$$

and so $f_j \to f$ in the $\|\cdot\|_{\mathbf{c}}$ norm.

1.5 We show that $C^{0,\gamma}(\overline{\Omega})$ is a Banach space; the case $C^{r,\gamma}$ then follows easily. If the sequence $\{f_j\}$ is Cauchy in $C^{0,\gamma}(\overline{\Omega})$ then given $\epsilon > 0$ there exists an N such that for $j, k \geq N$ we have

$$\|f_j - f_k\|_\infty + \sup_{x,y\in\overline{\Omega}} \frac{|[f_j(x) - f_k(x)] - [f_j(y) - f_k(y)]|}{|x-y|^\gamma} \leq \epsilon.$$

Because $C^0(\overline{\Omega})$ is complete we know that f_j converges to some $f \in C^0(\overline{\Omega})$. We just need to show that f is Hölder. However, since $f_k \to f$ uniformly we have

$$|[f_j(x) - f(x)] - [f_j(y) - f(y)]| \leq \epsilon|x-y|^\gamma$$

and so

$$\begin{aligned} |f(x) - f(y)| &\leq |f_j(x) - f_j(y)| + |[f_j(x) - f(x)] + [f_j(y) - f(y)]| \\ &\leq C_j|x-y|^\gamma + \epsilon|x-y|^\gamma, \end{aligned}$$

which shows that $f \in C^{0,\gamma}(\overline{\Omega})$.

1.6 If $f \in C^1(\overline{\Omega})$ then $|Df(x)|$ is uniformly bounded on $\overline{\Omega}$, by L, say. Since Ω is convex, given any two points $x, y \in \Omega$ the line segment joining x and y lies entirely in Ω. It follows that

$$\begin{aligned} |f(x) - f(y)| &= \left| \int_0^1 Df(y + \xi(x-y)) \cdot (x-y)\, d\xi \right| \\ &\leq L|x-y|, \end{aligned}$$

so f is Lipschitz.

1.7 We have

$$\begin{aligned} |u_h(x) - u_h(y)| &= \left| h^{-m} \int_\Omega \left[\rho\left(\frac{x-z}{h}\right) - \rho\left(\frac{y-z}{h}\right) \right] u(z)\, dz \right| \\ &\leq h^{-m} \int_\Omega \rho\left(\frac{x-z}{h}\right) |u(z) - u(z+y-x)|\, dz \\ &\leq C|y-x|^\gamma \end{aligned}$$

by using (1.7) so that u_h is also Hölder.

1.8 We prove the result by induction, supposing that it is true for $n = k$. Then for $n = k+1$ we take p such that

$$\left(\sum_{j=1}^{k-1} \frac{1}{p_j} \right) + \frac{1}{p} = 1,$$

to obtain

$$\int_\Omega |f_1(x)\cdots f_{k+1}(x)|dx \le \|f_1\|_{L^{p_1}}\cdots\|f_{k-1}\|_{L^{p_{k-1}}}\|f_k f_{k+1}\|_{L^p}. \tag{S1.2}$$

Now, we use the standard Hölder inequality, noting that

$$1 = \frac{p}{p_k} + \frac{p}{p_{k+1}};$$

thus

$$\int_\Omega (f_k f_{k+1})^p\,dx \le \left(\int_\Omega f_k^{p_k}\,dx\right)^{p/p_k}\left(\int_\Omega f_{k+1}^{p_{k+1}}\,dx\right)^{p/p_{k+1}},$$

and so

$$\|f_k f_{k+1}\|_{L^p} \le \|f_k\|_{L^{p_k}}\|f_{k+1}\|_{L^{p_{k+1}}},$$

which combined with (S1.2) gives (1.31) for $n = k+1$. Since the standard Hölder inequality is (1.31) for $n = 2$ the result follows.

1.9 Write

$$\int_\Omega |u(x)|^p\,dx = \int_\Omega |u(x)|^{q(r-p)/(r-q)}|u(x)|^{r(p-q)/(r-q)}\,dx.$$

Now note that

$$\frac{r-p}{r-q} + \frac{p-q}{r-q} = 1,$$

and so using Hölder's inequality we have

$$\int_\Omega |u(x)|^p\,dx \le \left(\int_\Omega |u(x)|^q\,dx\right)^{(r-p)/(r-q)}\left(\int_\Omega |u(x)|^r\,dx\right)^{(p-q)/(r-q)},$$

which becomes

$$\|u\|_{L^p} \le \|u\|_{L^q}^{q(r-p)/p(r-q)}\|u\|_{L^r}^{r(p-q)/p(r-q)},$$

as required.

1.10 If $s \in S(\Omega)$ then it is of the form of (1.10),

$$s(x) = \sum_{j=1}^n c_j\chi[I_j](x),$$

where the I_j are m-dimensional cuboids, each of the form

$$I = \prod_{k=1}^{m} [a_k, b_k].$$

It clearly suffices to approximate $\chi[I]$ to within ϵ in the L^p norm using an element of $C_c^0(\Omega)$. To do this, consider the function

$$\chi_\eta = \prod_{k=1}^{m} \phi_\eta(x_k; a_k, b_k),$$

where

$$\phi_\eta(x; b, a) = \begin{cases} (x-a)/\eta, & a \le x \le a+\eta, \\ 1, & a+\eta < x < b-\eta, \\ (b-x)/\eta, & b-\eta \le x \le b. \end{cases}$$

Clearly $\chi_\eta \in C_c^0(\Omega)$ and converges to $\chi[I]$ in $L^p(\Omega)$ as $\eta \to 0$.

1.11 Since $|g(x)| \le \|g\|_\infty$ almost everywhere, it follows that

$$|f(x)g(x)| \le |f(x)|\|g\|_\infty$$

almost everywhere, and so

$$\int_\Omega |f(x)g(x)|\,dx \le \int_\Omega |f(x)|\|g\|_\infty\,dx \le \|f\|_{L^1}\|g\|_\infty,$$

as claimed.

1.12 Since $\{x^{(n)}\}$ is Cauchy, given $\epsilon > 0$ there exists an N such that

$$\left\|x^{(n)} - x^{(m)}\right\|_{l^\infty} \le \epsilon \qquad \text{for all} \qquad n, m \ge N.$$

This implies that

$$\left|x_j^{(n)} - x_j^{(m)}\right| \le \epsilon \qquad \text{for all} \qquad n, m \ge N. \tag{S1.3}$$

In particular, we have $x_j^{(n)}$ is Cauchy for each j. So $x_j^{(n)} \to x_j$ as $n \to \infty$. It is then clear that $x = \{x_j\} \in l^\infty$, and taking the limit $m \to \infty$ in (S1.3) shows that

$$\left|x_j^{(n)} - x_j\right| \le \epsilon \qquad \text{for all} \qquad n \ge N, \qquad \text{for all} \qquad j.$$

It follows that $x^{(n)} \to x$ in l^∞, and so l^∞ is complete.

1.13 We know that the norm is positive definite, and so

$$\|x+\lambda y\|^2 = (x+\lambda y, x+\lambda y) = \|x\|^2 + 2\lambda(x, y) + \lambda^2\|y\|^2 \geq 0.$$

In particular, the quadratic equation for λ,

$$\lambda^2\|y\|^2 + 2\lambda(x, y) + \|x\|^2 = 0,$$

can have only one distinct real root. Therefore the discriminant "$b^2 - 4ac$" cannot be positive (which would give two real roots). In other words,

$$4(x, y)^2 - 4\|y\|^2\|x\|^2 \leq 0$$

or

$$|(x, y)| \leq \|x\|\|y\|,$$

which is the Cauchy–Schwarz inequality. We can now write

$$\begin{aligned}\|x+y\|^2 &= \|x\|^2 + 2(x, y) + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2,\end{aligned}$$

giving the triangle inequality.

1.14 We simply expand the left-hand side,

$$\begin{aligned}\|u+v\|^2 + \|u-v\|^2 &= \|u\|^2 + 2(u, v) + \|v\|^2 + \|u\|^2 - 2(u, v) + \|v\|^2 \\ &= 2\|u\|^2 + 2\|v\|^2,\end{aligned}$$

as required.

1.15 If $\{u_j\}$ is a dense subset of $l^2(\Gamma)$ then for each element $\gamma \in \Gamma$ there must exist a u_j that is within ϵ of 1 at γ and within ϵ of 0 for all other elements of Γ. Each such u_j is distinct. It follows that if Γ is uncountable then so are the $\{u_j\}$, and so $l^2(\Gamma)$ cannot be separable.

Chapter 2

2.1 We can apply the contraction mapping theorem to h^n to deduce that h^n has a unique fixed point x^*,

$$h^n(x^*) = x^*.$$

If we apply h to both sides then

$$h(h^n(x^*)) = h^{n+1}(x^*) = h^n(h(x^*)) \quad = \quad h(x^*),$$

showing that $h(x^*)$ is also a fixed point of h^n. Since the contraction mapping theorem guarantees that the fixed point of h^n is unique, we must have $h(x^*) = h^*$, and so h^* is also a fixed point of h.

2.2 The interval $[1, \infty)$ is closed but not compact, and the map $h : [1, \infty) \to [1, \infty)$ given by $x \mapsto x + 1/x$ satisfies

$$\begin{aligned} |h(x) - h(y)| &= |x - y|(1 - (xy)^{-1}) \\ &< |x - y| \end{aligned}$$

but clearly has no fixed point.

However, if X is compact and $h : X \to X$ satisfies

$$\|h(x) - h(y)\| < \|x - y\|, \tag{S2.1}$$

suppose that h has no fixed point. Then

$$\|h(x) - x\| > 0 \qquad \text{for all} \qquad x \in X,$$

and since $\|h(x) - x\|$ is continuous from X into $\mathbb{R}$ it obtains its lower bound, so that

$$\|h(x) - x\| \geq \epsilon \qquad \text{for all} \qquad x \in X,$$

and there exists some $y \in X$ such that $\|h(y) - y\| = \epsilon$. However, if we take $z = h(y)$ then from (S2.1) we have

$$\|h(z) - z\| < \epsilon,$$

a contradiction. So h has at least one fixed point. Uniqueness follows as in the proof of the standard contraction mapping theorem.

2.3 Take $\epsilon_n = 2^{-n}$ and apply the result of Exercise 1.2 so that there exists finite set $\{x_j^{(k)}\}$, $1 \leq j \leq M_k$, such that $|x - x_j^{(k)}| \leq 2^{-k}$. Set $N_k = \sum_{j=1}^{k} M_j$, and let $\{x_j\}$ be the sequence

$$x_1^{(1)}, \ldots, x_{M_1}^{(1)}, x_1^{(2)}, \ldots, x_{M_2}^{(2)}, x_1^{(3)}, \ldots.$$

2.4 Suppose that there are solutions $x_n(t)$ of

$$dx/dt = f(x) \qquad \text{with} \qquad x(0) = x_0 \tag{S2.2}$$

such that $x_n(\tau) \to x^*$. We need to show that there is a solution of (S2.2) with $x(\tau) = x^*$. Now, if f is bounded then the sequence $x_n(t)$ satisfies

$$\sup_{t\in[0,\tau]} |x_n(t)| \le |x_0| + \tau\|f\|_\infty \quad \text{and} \quad |x_n(t) - x_n(s)| \le \|f\|_\infty |t-s|,$$

the conditions of the Arzelà–Ascoli theorem (Theorem 2.5). It follows that there is a subsequence that converges uniformly on $[0, \tau]$, and as in the proof of Theorem 2.6 the limit $x(t)$ satisfies (S2.2). Since $x_n \to x$ uniformly on $[0, \tau]$, in particular we have $x(\tau) = x^*$ as required.

2.5 When $|x| \neq 0$ then it follows that

$$\frac{d}{dt}|x|^2 = 2|x|\frac{d}{dt}|x|,$$

and (2.27) follows immediately. When $|x(t_0)| = 0$, since $C(t)$ is continuous, for any $\epsilon > 0$ we have

$$\tfrac{1}{2}\frac{d}{dt}|x|^2 \le [C(t_0) + \epsilon]|x|$$

for $t - t_0$ small enough, and so it follows from Lemma 2.7 that

$$|x(t)|^2 \le \big([C(t_0) + \epsilon](t - t_0)\big)^2.$$

Therefore

$$|x(t+h)| \le [C(t_0) + \epsilon]t,$$

and so

$$\frac{d}{dt_+}|x| \le C(t_0) + \epsilon.$$

Since this holds for any $\epsilon > 0$ we have (2.27).

2.6 If

$$y(t) = \int_0^t b(s)x(s)\,ds$$

then

$$\frac{dy}{dt} = b(t)x(t) \le a(t)b(t) + b(t)y(t),$$

and so

$$\left(\frac{dy}{dt} - b(t)y(t)\right)\exp\left(-\int_0^t b(s)\,ds\right) \le a(t)b(t)\exp\left(-\int_0^t b(s)\,ds\right).$$

If $a(t)$ is increasing then we can replace $a(t)$ on $[0, T]$ with $a(T)$, and so

$$\frac{d}{dt}\left[y(t)\exp\left(-\int_0^t b(s)\,ds\right)\right] \leq a(T)b(t)\exp\left(-\int_0^t b(s)\,ds\right).$$

Integrating both sides between 0 and T gives us

$$y(T)\exp\left(-\int_0^T b(s)\,ds\right) \leq a(T)\int_0^T b(t)\exp\left(-\int_0^t b(s)\,ds\right)dt$$

and so

$$y(T) \leq a(T)\int_0^T b(t)\exp\left(\int_t^T b(s)\,ds\right)dt.$$

We can integrate the right-hand side to obtain

$$y(T) \leq a(T)\left[\exp\left(\int_0^T b(s)\,ds\right) - 1\right],$$

and so, using (2.28), we have

$$x(T) \leq a(T)\exp\left(\int_0^T b(s)\,ds\right)$$

as claimed.

2.7 As in the proof of Proposition 2.10 we consider the difference of two solutions, $z(t) = x(t) - y(t)$, which satisfies

$$\begin{aligned}\frac{dz}{dt} &= f(x) - g(y)\\ &= f(x) - f(y) + f(y) - g(y).\end{aligned}$$

We now use Lemma 2.9 to deduce that

$$\begin{aligned}\frac{d}{dt_+}|z| &\leq |f(x) - f(y)| + |f(y) - g(y)|\\ &\leq L|z| + \|f - g\|_\infty.\end{aligned}$$

An application of Gronwall's inequality [(2.21) in Lemma 2.8] now yields (2.29).

Chapter 3

3.1 We denote

$$\|A\|_1 = \{\text{smallest } M \text{ such that } \|Ax\|_Y \leq M\|x\|_X \text{ for all } x \in X\}$$

and

$$\|A\|_2 = \sup_{\|x\|_X=1} \|Ax\|_Y.$$

First we take $x \neq 0$ and put $y = x/\|x\|_X$; then we have

$$\|Ay\|_Y \leq \|A\|_2 \qquad \Rightarrow \qquad \|Ax\|_Y \leq \|A\|_2\|x\|_X$$

for all $x \in X$, and so $\|A\|_1 \leq \|A\|_2$. Furthermore, it is clear that, for any M,

$$\|Ax\|_Y \leq M\|x\|_X \qquad \text{for all} \qquad x \in X \qquad \Rightarrow \qquad \|A\|_2 \leq M,$$

and so $\|A\|_2 \leq \|A\|_1$. Thus $\|A\|_1 = \|A\|_2$.

3.2 I is clearly bounded from $C^0([O, L])$ into itself, since

$$\|I(f)\|_\infty \leq L\|f\|_\infty.$$

For the L^2 bound, first observe, by using the Cauchy–Schwarz inequality, that $I(f)(x)$ is defined for all x if $f \in L^2$. Then

$$\begin{aligned}
|I(f)|^2 &= \int_0^L |I(f)(x)|^2\,dx \\
&= \int_0^L \left(\int_0^x f(s)\,ds\right)^2 dx \\
&= \int_0^L \left(\int_0^x ds\right)\left(\int_0^x |f(s)|^2\,ds\right) dx \\
&\leq L^2 \int_0^L |f(s)|^2\,ds \\
&\leq L^2|f|^2.
\end{aligned}$$

Thus I is a bounded operator on both spaces.

3.3 Suppose that $A^{-1}y_1 = x_1$ and that $A^{-1}y_2 = x_2$. Then it is clear that

$$A(x_1 + x_2) = y_1 + y_2.$$

Since the inverse is unique it follows that

$$A^{-1}(y_1 + y_2) = A^{-1}y_1 + A^{-1}y_2.$$

3.4 For each $x \in X$, $P_n x$ converges to x, and so it follows that the sequence $\{P_n x\}_{n=1}^{\infty}$ is bounded:

$$\sup_{n \in \mathbb{Z}^+} \|P_n x\|_X < \infty$$

for each $x \in X$. From the principle of uniform boundedness (Theorem 3.7) we immediately obtain

$$\sup_{n \in \mathbb{Z}^+} \|P_n\|_{\mathrm{op}} < \infty,$$

as claimed.

3.5 It is clear that $\phi_i(x)\phi_j(y)$ is an element of $L^2(\Omega \times \Omega)$ and that

$$\int_{\Omega \times \Omega} [\phi_i(x)\phi_j(y)][\phi_k(x)\phi_l(y)]\, dx\, dy = \delta_{ik}\delta_{jl},$$

and so they certainly form an orthonormal set. If $k \in L^2(\Omega \times \Omega)$ then $k(\cdot, y) \in L^2(\Omega)$, and we can write

$$k(x, y) = \sum_{i=1}^{\infty} u_i(y)\phi_i(x),$$

where

$$u_i(y) = \int_{\Omega} k(x, y)\phi_i(x)\, dx.$$

Since

$$\begin{aligned}
\int_{\Omega} |u_i(y)|^2\, dy &= \int_{\Omega} \left| \int_{\Omega} k(x, y)\phi_i(x)\, dx \right|^2 dy \\
&\le \int_{\Omega} \left(\int_{\Omega} |k(x, y)|^2\, dx \int_{\Omega} |\phi_i(x)|^2\, dx \right) dy \\
&\le \int_{\Omega \times \Omega} |k(x, y)|^2\, dx\, dy,
\end{aligned}$$

we have $u_i \in L^2(\Omega)$. So we can write

$$u_i(y) = \sum_{j=1}^{\infty} \left(\int_{\Omega} u_i(y)\phi_j(y)\, dy \right) \phi_j(y),$$

which yields the expression

$$k(x, y) = \sum_{i,j=1}^{\infty} \left(\int_{\Omega \times \Omega} k(x, y)\phi_i(x)\phi_j(y)\, dx\, dy \right) \phi_i(x)\phi_j(y),$$

as claimed.

3.6 We consider the approximations to A given by the truncated sums,

$$A_n u = \sum_{j=1}^{n} \lambda_j (u, w_j) w_j.$$

Using Lemma 3.12 we see that each operator A_n is compact. We now want to show that

$$\|A - A_n\|_{\text{op}} \to 0,$$

and it then follows from Theorem 3.10 that A is compact. However, this convergence is clear, since

$$\begin{aligned}
\|(A - A_n)u\| &= \left\| \sum_{j=n+1}^{\infty} \lambda_j (u, w_j) w_j \right\| \\
&\le \lambda_{n+1} \left\| \sum_{j=n+1}^{\infty} (u, w_j) w_j \right\| \\
&\le \lambda_{n+1} \left(\sum_{j=n+1}^{\infty} |(u, w_j)|^2 \right)^{1/2} \\
&\le \lambda_{n+1} \|u\|,
\end{aligned}$$

and $\lambda_{n+1} \to 0$ as $n \to \infty$. Thus A is compact. That A is symmetric follows by taking the inner product of Au with v to give

$$(Au, v) = \sum_{j=1}^{\infty} \lambda_j (u, w_j)(v, w_j) = (u, Av).$$

3.7 We know from Lemma 3.4 that A^{-1} exists iff $\text{Ker}(A) = 0$. So we show that if $Ax = 0$ then $x = 0$. Because A is bounded below we have

$$0 = \|Ax\|_Y \ge k \|x\|_X,$$

so that $\|x\|_X = 0$. For $y \in R(A)$ we can use the lower bound on A to deduce that

$$\|A^{-1} y\|_X \le \frac{1}{k} \|AA^{-1} y\|_Y = \frac{1}{k} \|y\|_Y,$$

so that A^{-1} is bounded.

3.8 Since G is a solution of the homogeneous equation on both sides of $x = y$, we must have

$$G(x, y) = \begin{cases} C_1(y)u_1(x), & a \le x < y, \\ C_2(y)u_2(x), & y \le x \le b. \end{cases}$$

The conditions at y require

$$C_1(y)u_1(y) = C_2(y)u_2(y),$$
$$C_1(y)u_1'(y) + p(y)^{-1} = C_2(y)u_2'(y).$$

Solving these simultaneous equations for C_1 and C_2 gives

$$C_1(y) = u_2(y)/W_p(y) \qquad \text{and} \qquad C_2(y) = u_1(y)/W_p(y),$$

where

$$W_p(y) = p(y)[u_1(y)u_2'(y) - u_2(y)u_1'(y)].$$

Differentiating W_p with respect to y and cancelling the $pu_1'u_2'$ terms gives

$$W_p' = p'(u_1u_2' - u_2u_1') + p[u_1u_2'' - u_2u_1''].$$

If we use the differential equation $L[u_1] = L[u_2] = 0$ to substitute for the terms pu_1'' and pu_2'' we see that in fact $W_p' = 0$, so that W_p is a constant. We therefore obtain (3.28), and $G(x, y)$ is symmetric.

3.9 Proposition 3.13 and Lemma 3.16 show that the integral operator K defined by

$$[Ku](x) = \int_\Omega k(x, y)u(y)\,dy$$

is a compact symmetric mapping from $L^2(\Omega)$ into $L^2(\Omega)$. It follows from Theorem 3.18 that K has a set of eigenfunctions $u_n(x)$ with corresponding eigenvalues λ_n, so that $Ku_n = \lambda_n u_n$:

$$\int_\Omega k(x, y)u_n(y)\,dy = \lambda_n u_n(x).$$

Since $\lambda_j \neq 0$ for all j there is no nonzero u such that $Ku = 0$. In this case $\text{Ker}K = \{0\}$, and so we can expand any $f \in L^2(\Omega)$ in terms of the eigenfunctions of K,

$$f = \sum_{j=1}^{\infty} (f, u_j) u_j.$$

It is now easy to see that the solution of (3.29) is given by

$$u(x) = \sum_{j=1}^{\infty} \frac{(f, u_j)}{\lambda_j} u_j(x),$$

as claimed.

3.10 We have

$$A^{-\alpha} w_j = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-\lambda_j t} \, dt \; w_j. \tag{S3.1}$$

Now,

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \, dt,$$

and so, substituting $u = \lambda_j t$ in (S3.1), we have

$$\int_0^{\infty} \lambda_j^{1-\alpha} u^{\alpha-1} e^{-u} \frac{du}{\lambda_j} = \lambda_j^{-\alpha} \Gamma(\alpha),$$

which gives

$$A^{-\alpha} w_j = \lambda_j^{-\alpha} w_j$$

as required. Since $A^{-\alpha}$ is characterised by its action on the eigenfunctions the two expressions are equivalent.

3.11 We have

$$\begin{aligned}
\|A^s u\|^2 &= \sum_{j=1}^{\infty} \lambda_j^{2s} |c_j|^2 \\
&= \sum_{j=1}^{\infty} \lambda_j^{2(s-\alpha)} |c_j|^{2\varphi} \lambda_j^{2\alpha} |c_j|^{2(1-\varphi)} \\
&\leq \left(\sum_{j=1}^{\infty} \lambda_j^{2(s-\alpha)/\varphi} |c_j|^2 \right)^{\varphi} \left(\sum_{j=1}^{\infty} \lambda_j^{2\alpha/(1-\varphi)} |c_j|^2 \right)^{1-\varphi} \\
&\leq \left\| A^{(s-\alpha)/\varphi} u \right\|^{2\varphi} \left\| A^{\alpha/(1-\varphi)} u \right\|^{2(1-\varphi)},
\end{aligned}$$

which gives the result on setting $\varphi = (k-s)/(k-l)$ and $\alpha = k(s-l)/(k-l)$.

3.12 Take $x = \sum_{j=1}^{\infty} x_j w_j$ and consider the series expansion

$$\frac{(e^{-Ah} - I)}{h} x + Ax = \sum_{j=1}^{\infty} \left[\frac{(e^{-\lambda_j h} - 1)}{h} + \lambda_j \right] x_j w_j. \qquad \text{(S3.2)}$$

Observe that

$$\sum_{j=n+1}^{\infty} \left[\frac{(e^{-\lambda_j h} - 1)}{h} + \lambda_j \right] x_j w_j = \sum_{j=n+1}^{\infty} \left[\frac{(e^{-\lambda_j h} - 1)}{\lambda_j h} + 1 \right] \lambda_j x_j w_j.$$

The mean-value theorem tells us that $(e^{-z} - 1)/z \leq 1$, and so

$$\left| \sum_{j=n+1}^{\infty} \left[\frac{(e^{-\lambda_j h} - 1)}{h} + \lambda_j \right] x_j w_j \right|^2 \leq 4 \sum_{j=n+1}^{\infty} \lambda_j^2 |x_j|^2, \qquad \text{(S3.3)}$$

which tends to zero as $n \to \infty$.

It follows that given $\epsilon > 0$ we can choose an n such that the infinite sum in (S3.3) is bounded above by $\epsilon/2$. It is then clear that the finite sum

$$\sum_{j=1}^{n} \left[\frac{(e^{-\lambda_j h} - 1)}{h} + \lambda_j \right] x_j w_j$$

converges to zero as $h \to 0$, and so for small enough h the whole expression in (S3.2) is bounded by ϵ, as required.

Chapter 4

4.1 Let $P = \{\text{orthonormal subsets of } H\}$, and define an order on P such that $a \leq b$ if $a \subseteq b$. If $\{C_i\}$ is a chain ($i \in \mathcal{I}$) then $\mathcal{C} = \cup_i C_i$ is an upper bound. Zorn's lemma implies that there is a maximal orthonormal set $\{e_i\}_{i \in I}$. The argument of the second part of Proposition 1.23 now shows that the $\{e_i\}$ form a basis.

4.2 Take $z \notin Y$. Then if w is contained in the linear span of z and Y it has a unique decomposition of the form

$$w = y + \alpha z \qquad \text{with} \qquad y \in Y,$$

as in the proof of the Hahn–Banach theorem. We can therefore define a nonzero linear functional on the linear span of z and Y via

$$f(y + \alpha z) = \alpha.$$

The functional f is zero on Y, and we can extend it to a nonzero linear functional on X by using the Hahn–Banach theorem.

4.3 It is immediate from Hölder's inequality that

$$|L_f(g)| \le \|f\|_{L^\infty}\|g\|_{L^1}$$

and so

$$\|L_f\|_{(L^1)^*} \le \|f\|_{L^\infty}. \tag{S4.1}$$

To show equality consider the sequence of functions

$$g_p(x) = |f(x)|^{p-2}f(x).$$

Since $f \in L^\infty(\Omega)$ and Ω is bounded we have $g_p(x) \in L^1(\Omega)$ for every p, with

$$\|g_p\|_{L^1} = \|f\|_{L^{p-1}}^{p-1}.$$

It follows from

$$|L_f(g_p)| = \|f\|_{L^p}^p$$

that

$$\|L_f\|_{(L^1)^*} \ge \frac{\|f\|_{L^p}^p}{\|f\|_{L^{p-1}}^{p-1}}.$$

Since $f \in L^\infty$ we can use the result of Proposition 1.16,

$$\|f\|_{L^\infty} = \lim_{p\to\infty}\|f\|_{L^p},$$

to deduce that

$$\|L_f\|_{(L^1)^*} \ge \|f\|_{L^\infty},$$

which combined with (S4.1) gives the required equality.

4.4 Since M is a linear subspace of H it is also a Hilbert space. The Riesz theorem then shows that given a linear functional f on M there exists an $m \in M$ such that

$$f(x) = (m, x) \qquad \text{for all} \qquad x \in M.$$

Now define F on H by

$$F(u) = (m, u);$$

it is clear that F is an extension of F and that $\|F\| = \|f\|$.

4.5 If $x \notin M$ then the argument of Solution 4.2 shows that there exists an element $f \in X^*$ with $f|_M = 0$ but $f(x) \ne 0$. So if $f(x) = 0$ for all such f then we must have $x \in M$. Now if $x_n \rightharpoonup x$ then for each $f \in X^*$ with $f|_M = 0$ we have

$$f(x) = \lim_{n\to\infty} f(x_n) = 0,$$

and so it follows that $x \in M$.

The linear span of the $\{x_n\}$ forms a linear subspace M of X, and clearly $x_n \in M$ for each n. It follows that x is contained in the linear span of the $\{x_n\}$ and so can be written in the form

$$x = \sum_{j=1}^{\infty} c_j x_j. \tag{S4.2}$$

[In fact x can be written as a convex combination of the $\{x_j\}$, that is, (S4.2) with $c_j \geq 0$ and $\sum_j c_j = 1$; see Yosida (1980, p. 120).]

4.6 For any $t \in [a, b]$,

$$\delta_t : x \mapsto x(t)$$

is a bounded linear functional on $C^0([a, b])$. Since $x_n \rightharpoonup x$, we have

$$\delta_t(x_n) \to \delta_t(x),$$

and so $x_n(t) \to x(t)$ for each $t \in [a, b]$.

4.7 Write

$$\|x_n - x\|^2 = \|x_n\|^2 + \|x\|^2 - 2(x, x_n),$$

and then take limits on the right-hand side, using norm convergence on $\|x_n\|^2$ and weak convergence on (x, x_n), to show that

$$\lim_{n\to\infty} \|x_n - x\|^2 = 0,$$

which is $x_n \to x$.

Chapter 5

5.1 Simply write

$$\langle D^\alpha u, \phi_n \rangle = (-1)^{|\alpha|} \langle u, D^\alpha \phi_n \rangle,$$

and then using the definition of convergence in $\mathcal{D}(\Omega)$ (Definition 5.2) we have $D^\alpha \phi_n \to D^\alpha \phi$ in $\mathcal{D}(\Omega)$, and so

$$\begin{aligned}\langle D^\alpha u, \phi_n\rangle &\to (-1)^{|\alpha|}\langle u, D^\alpha \phi\rangle \\ &= \langle D^\alpha u, \phi\rangle,\end{aligned}$$

so that $D^\alpha u$ is indeed a distribution.

5.2 If $\phi_n \in \mathcal{D}(\Omega)$ with $\phi_n \to \phi$ in $\mathcal{D}(\Omega)$ then $\psi\phi_n \to \psi\phi$ in $\mathcal{D}(\Omega)$. It follows that

$$\langle \psi u, \phi_n\rangle = \langle u, \psi\phi_n\rangle \to \langle u, \psi\phi_n\rangle = \langle \psi u, \phi\rangle,$$

and so $\psi u \in \mathcal{D}'(\Omega)$.

Given $\phi \in \mathcal{D}(\Omega)$ we have

$$\begin{aligned}\langle D(\psi u), \phi\rangle &= -\langle \psi u, \phi'\rangle \\ &= -\langle u, \psi\phi'\rangle \\ &= -\langle u, \psi\phi' + \phi\psi'\rangle + \langle u, \phi\psi'\rangle \\ &= \langle Du, \psi\phi\rangle + \langle uD\psi, \phi\rangle \\ &= \langle \psi Du + uD\psi, \phi\rangle,\end{aligned}$$

as claimed.

5.3 Assume that $|f_n| \le M$ for every n. For every $\phi \in C_c^\infty(\Omega)$ we know that

$$\int_\Omega f_n\phi\, dx \tag{S5.1}$$

is a Cauchy sequence. Since $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$ (Corollary 1.14), for each $u \in L^2(\Omega)$ we can find a sequence of $\phi_n \in C_c^\infty(\Omega)$ with $\phi_n \to u$ in $L^2(\Omega)$. Then, given $\epsilon > 0$, choose K such that

$$|\phi_k - u| \le \epsilon/4M \qquad \text{for all} \qquad k \ge K$$

and then choose N such that

$$\left|\int_\Omega (f_n - f_m)\phi_k\, dx\right| \le \epsilon/2 \qquad \text{for all} \qquad n, m \ge N.$$

It follows that for all $n, m \ge N$

$$\begin{aligned}\left|\int_\Omega (f_n - f_m)u\, dx\right| &\le |f_n - f_m||u - \phi_k| + \epsilon/2 \\ &\le (2M)(\epsilon/4M) + \epsilon/2 = \epsilon,\end{aligned}$$

and so (S5.1) is a Cauchy sequence for every $u \in L^2(\Omega)$, showing that $f_n \rightharpoonup f$ in $L^2(\Omega)$.

5.4 Suppose that the result is true for $k = n$. We show that it holds for $k = n + 1$, which then gives a proof by induction since the statement of Proposition 5.8 gives (5.45) for $k = 1$. We know that

$$\|u\|_{H^{n+1}}^2 = \|u\|_{H^n}^2 + \sum_{|\alpha|=n+1} |D^\alpha u|^2,$$

which along with the induction hypothesis becomes

$$\|u\|_{H^{n+1}}^2 \le C(n) \sum_{|\alpha|=n} |D^\alpha u|^2 + \sum_{|\alpha|=n+1} |D^\alpha u|^2. \tag{S5.2}$$

We therefore consider $|D^\alpha u|$ for $|\alpha| = n$. Since $u \in H_0^{n+1}(\Omega)$ we must have $D^\alpha u \in H_0^1(\Omega)$, and so

$$|D^\alpha u| \le C|D_1 D^\alpha u| = C|D^\beta u|$$

with $|\beta| = n + 1$, by using (5.11) from the proof of Proposition 5.8. It follows from (S5.2) that

$$\|u\|_{H^{n+1}}^2 \le C(n+1) \sum_{|\alpha|=n+1} |D^\alpha u|^2,$$

which is the result for $k = n + 1$.

5.5 Consider a sequence of $u_n \in C^\infty(\overline{\Omega})$ that approximates u in $H^k(\Omega)$. Then the derivatives of ψu_n are given by the Leibniz formula (1.6)

$$D^\alpha(\psi u_n) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \psi D^{\alpha-\beta} u_n,$$

and so

$$\begin{aligned} |D^\alpha(\psi u_n)| &\le \sum_{\beta \le \alpha} \binom{\alpha}{\beta} |D^\beta \psi||D^{\alpha-\beta} u_n| \\ &\le \left(\sum_{\beta \le \alpha} \binom{\alpha}{\beta} |D^\beta \psi| \right) \|u_n\|_{H^k}. \end{aligned}$$

In this way the derivatives up to and including order k are bounded in L^2 by a constant (depending on ψ) times the H^k norm of u_n, and so

$$\|\psi u_n\|_{H^k} \le C(\psi)\|u_n\|_{H^k}.$$

It follows that ψu_n is Cauchy in $H^k(\Omega)$, and so in the limit as $n \to \infty$ we have $\psi u \in H^k(\Omega)$ with

$$\|\psi u\|_{H^k} \le C(\psi)\|u\|_{H^k}$$

as required.

5.6 First we show that $u \in L^2(B(0,1))$:

$$\int_{B(0,1)} \left[\log\log\left(1+\frac{1}{|x|}\right)\right]^2 dx\,dy = \int_0^{2\pi}\int_0^1 r\log\log(1+1/r)\,dr\,d\theta,$$

which is finite since the integrand is bounded. Now, since

$$\frac{\partial u}{\partial x} = \frac{1}{\log(1+1/|x|)}\frac{x}{|x|^2(1+|x|)}$$

we have

$$\begin{aligned}\int_{B(0,1)} |\nabla u(x)|^2\,dx\,dy \\ &= \int_{B(0,1)} \frac{1}{\log(1+1/|x|)^2}\frac{1}{|x|^2(1+|x|)^2}\,dx\,dy \\ &= \int_0^{2\pi}\int_0^1 \frac{1}{\log(1+1/r)^2}\frac{1}{r(1+r)^2}\,dr. \qquad \text{(S5.3)}\end{aligned}$$

If we make the substitution $u = 1/r$ this becomes

$$\int_1^\infty \frac{1}{u+(1/u)}\frac{1}{\log(1+u)^2}\,du.$$

This integral is bounded by

$$\int_1^\infty \frac{1}{u(\log u)^2}\,du,$$

and since the integrand is the derivative of $-1/\log u$ it follows that the integral in (S5.3) is finite. Therefore $u \in H^1(B(0,1))$, even though it is unbounded.

5.7 First integrate (5.46) with respect to x_1, so that

$$\begin{aligned}\int_{-\infty}^\infty |u(x)|^3\,dx_1 &\le 6\left(\int_{-\infty}^\infty uD_1u\,dy_1\right)^{1/2}\int_{-\infty}^\infty\left(\int_{-\infty}^\infty uD_2u\,dy_2\right)^{1/2} \\ &\quad\times\left(\int_{-\infty}^\infty uD_3u\,dy_3\right)^{1/2} dx_1 \\ &\le 6\left(\int_{-\infty}^\infty uD_1u\,dy_1\right)^{1/2}\left(\iint_{-\infty}^\infty uD_2u\,dx_1\,dy_2\right)^{1/2} \\ &\quad\times\left(\iint_{-\infty}^\infty uD_3u\,dx_1\,dy_3\right)^{1/2}.\end{aligned}$$

Now integrate with respect to x_2 to obtain

$$\iint_{-\infty}^{\infty} |u(x)|^3 \, dx_1 \, dx_2 \leq 6\left(\iint_{-\infty}^{\infty} u D_2 u \, dx_1 \, dy_2 \right)^{1/2} \times \left(\iint_{-\infty}^{\infty} u D_1 u \, dy_1 \, dx_2 \right)^{1/2} \left(\iiint_{-\infty}^{\infty} u D_3 u \, dx_1 \, dx_2 \, dy_3 \right)^{1/2}.$$

Finally, integrating with respect to x_3 gives

$$\int_\Omega |u(x)|^3 \, dx \leq 6 \prod_{j=1}^{3} \left(\int_\Omega u D_j u \, dx \right)^{1/2},$$

and so

$$\|u\|_{L^3}^3 \leq C|u|^{3/2}|Du|^{3/2},$$

which gives

$$\|u\|_{L^3} \leq C|u|^{1/2}\|u\|_{H^1}^{1/2},$$

as required.

5.8 We simply apply the argument of Theorem 5.29 to the functions $v = D^\alpha u$ for each α with $|\alpha| \leq j$. It follows that $v \in H^{k-j}(\Omega)$, and since $k - j > m/2$ we can use Theorem 5.29 to deduce that $v \in C^0(\overline{\Omega})$ with

$$\|v\|_\infty \leq C\|u\|_{H^{k-j}} \leq C\|u\|_{H^k}.$$

Combining the estimates for each $|\alpha| \leq j$ shows that $u \in C^j(\overline{\Omega})$ with

$$\|u\|_{C^j(\overline{\Omega})} \leq C\|u\|_{H^k(\Omega)}$$

as claimed.

5.9 Suppose that the inequality does not hold. Then for each $k \in \mathbb{Z}^+$ there must exist $u_k \in V$ such that

$$|u_k| \geq k|\nabla u_k|. \tag{S5.4}$$

If we set $v_k = u_k/|u_k|$ so that $|v_k| = 1$, (S5.4) becomes

$$|\nabla v_k| \leq k^{-1}. \tag{S5.5}$$

It follows that v_k is a bounded sequence in $H^1(\Omega)$, and so using Theorem 5.32 it has a subsequence that converges in $L^2(\Omega)$ to some

$v \in V$ with

$$\int_\Omega v(x)\, dx = 0 \qquad \text{and} \qquad |v| = 1. \tag{S5.6}$$

However, we can also use (S5.5) along with the L^2 convergence of v_k to v to show that for any $\phi \in \mathcal{D}(\Omega)$ and any j

$$\int_\Omega v D^\alpha \phi\, dx = \lim_{k\to\infty} \int_\Omega v_k D_j \phi\, dx = -\lim_{k\to\infty} \int_\Omega D_j v_k \phi\, dx = 0.$$

It follows that $Dv = 0$, and so, using the hint, v is constant almost everywhere. This contradicts (S5.6), and so we have the inequality (5.47).

5.10 Suppose that $\{u_n\}$ is a bounded sequence in $L^2(\Omega)$. Then, since L^2 is reflexive, there is a subsequence that converges weakly in $L^2(\Omega)$, i.e. for every $\phi \in L^2(\Omega)$ we have

$$(u_n, \phi) \to (u, \phi)$$

for some $u \in L^2(\Omega)$. Now, suppose that u_n does not converge to u in $H^{-1}(\Omega)$, so that there exists an $\epsilon > 0$ such that, for some subsequence $\{u_n\}$,

$$\sup_{\{\phi\in H_0^1(\Omega):\|\phi\|_{H_0^1}=1\}} |(u_n - u, \phi)| \geq \epsilon.$$

Then there exist ϕ_n with $\|\phi_n\|_{H_0^1} = 1$ such that

$$|(u_n - u, \phi_n)| \geq \epsilon/2.$$

Since $\{\phi_n\}$ is a bounded sequence in $H_0^1(\Omega)$ and $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, there exists a subsequence that is convergent in $L^2(\Omega)$ to some ϕ. It follows (on relabelling) that

$$|(u_n - u, \phi)| \geq \epsilon/4$$

for n large enough. But this contradicts the weak convergence of u_n to u in L^2, and so we must have $u_n \to u$ in $H^{-1}(\Omega)$.

5.11 Since

$$u = \sum_{k\in\mathbb{Z}^m} c_k e^{2\pi i k\cdot x/L}$$

we have

$$Du = \sum_{k\in\mathbb{Z}^m} \frac{2\pi i k}{L} c_k e^{2\pi i k\cdot x/L}.$$

It follows that

$$|u|^2 = L^m \sum_{k\in\mathbb{Z}^m} |c_k|^2 \qquad \text{and} \qquad |Du|^2 = L^m \sum_{k\in\mathbb{Z}^m} (4\pi/L)^2 |k|^2 |c_k|^2,$$

and so

$$|u| \leq \left(\frac{L}{2\pi}\right)|Du|$$

as claimed.

Chapter 6

6.1 Start with

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f(x) v(x) \, dx,$$

and integrate the left-hand side by parts to give

$$\int_\Omega (\Delta u - f) v \, dx = 0.$$

Since $u \in C^2(\Omega)$ and $f \in C^0(\Omega)$, we have

$$\varphi \equiv \Delta u - f \in C^0(\Omega).$$

It therefore suffices to show that if

$$\int_\Omega \varphi v \, dx = 0 \qquad \text{for all} \qquad v \in C_c^1(\Omega)$$

then $\varphi = 0$. Suppose that $\varphi(x) \neq 0$ for some $x \in \Omega$. Then since φ is continuous there is a neighbourhood N of x on which $\varphi(x)$ is of constant sign. Taking a function v that is positive and has compact support within N implies that

$$\int_\Omega \varphi v \, dx = \int_N \varphi v \, dx \neq 0,$$

a contradiction. That u satisfies $u|_{\partial\Omega} = 0$ follows from $u \in H_0^1(\Omega) \cap C^0(\overline{\Omega})$, using Theorems 5.35 and 5.36.

6.2 Take the inner product of $Lu = f$ with a $v \in C_c^1(\Omega)$,

$$-\int_\Omega \sum_{i,j=1}^m \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)v(x)+\sum_{i=1}^m b_i(x)\frac{\partial u}{\partial x_i}v(x)+c(x)u(x)v(x)\,dx$$
$$= \int_\Omega f(x)v(x)\,dx,$$

and integrate the first term by parts,

$$\int_\Omega \sum_{i,j=1}^m a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{i=1}^m b_i(x)\frac{\partial u}{\partial x_i}v(x) + c(x)u(x)v(x)\,dx$$
$$= \int_\Omega f(x)v(x)\,dx.$$

We can now introduce a bilinear form

$$a(u,v)=\int_\Omega \sum_{i,j=1}^m a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{i=1}^m b_i(x)\frac{\partial u}{\partial x_i}v(x)+c(x)u(x)v(x)\,dx,$$

and write the equation as

$$a(u,v) = (f,v) \qquad \text{for all} \qquad v \in C_c^1(\Omega).$$

As before we use the density of $C_c^1(\Omega)$ in $H_0^1(\Omega)$ to generalise to $f \in H^{-1}(\Omega)$ and the weak form of the problem is thus to find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = \langle f,v\rangle \qquad \text{for all} \qquad v \in H_0^1(\Omega).$$

6.3 By definition

$$\begin{aligned} a(u,u) &= \sum_{i,j=1}^m \int_\Omega a_{ij}(x)D_j u D_i u\,dx + \sum_{i=1}^m \int_\Omega b_i(x)D_i u\, u\,dx \\ &\quad + \int_\Omega c(x)u^2\,dx \\ &\geq \theta \int_\Omega |\nabla u|^2\,dx - \max_i \|b_i\|_{L^\infty}\int_\Omega |\nabla u|\,|u|\,dx \\ &\quad - \|c\|_{L^\infty}\int_\Omega |u|^2\,dx. \end{aligned}$$

Now we use Young's inequality with ϵ,

$$ab \leq \epsilon \frac{a^2}{2} + \frac{b^2}{2\epsilon},$$

to split the second term,

$$a(u,u) \geq \tfrac{1}{2}\theta \int_\Omega |\nabla u|^2 \, dx - \left(\theta \max_i \|b_i\|_{L^\infty}\right)^{-1} \int_\Omega |u|^2 \, dx$$
$$- \|c\|_{L^\infty} \int_\Omega |u|^2 \, dx,$$

and so

$$a(u,u) \geq C\|u\|_{H_1}^2 - \lambda |u|^2,$$

as required.

6.4 Consider the bilinear form $b(u, v)$ corresponding to the operator $L + \alpha$. Then

$$b(u,v) = a(u,v) + \alpha \underbrace{(u,v)}_{L^2}$$

is a continuous bilinear form on H_0^1: clearly (u, v) is, and $a(u, v)$ is since

$$\begin{aligned} |a(u,v)| &\leq \sum_{i,j=1}^m \int_\Omega |a_{ij}||D_j u||D_i v| \, dx + \sum_{i=1}^m \int_\Omega |b_i||D_i u||v| \, dx \\ &\quad + \int_\Omega |c||u||v| \, dx \\ &\leq C\|u\|_{H^1}\|v\|_{H^1}. \end{aligned}$$

Furthermore, b satisfies the coercivity condition, since

$$\begin{aligned} b(u,u) &= a(u,u) + \alpha(u,u) \\ &\geq C\|u\|_{H_1}^2 - \lambda|u|^2 + \alpha|u|^2 \\ &\geq C\|u\|_{H_0^1}^2. \end{aligned}$$

We can now apply the Lax–Milgram lemma to obtain the conclusion.

6.5 First, it is easy to see that if (6.31) holds for all $v \in H^1(\Omega)$ then choosing $v = 1$ we have

$$\int_\Omega f(x) \, dx = 0.$$

We cannot immediately apply the Lax–Milgram lemma to the equation

$$a(u, v) = (f, v),$$

since

$$|a(u, u)| = |\nabla u|^2 = \|u\|_{H^1}^2 - |u|^2,$$

and so $a(u, v)$ is not coercive. To deal with the L^2 part we need a Poincaré-type inequality. Note that if $\int_\Omega f(x)\,dx = 0$, then

$$(f, v) = \left(f, v - \int_\Omega v(x)\,dx\right),$$

since subtracting the constant from v does not make any difference, and similarly

$$a(u, v) = a\left(u, v - \int_\Omega v(x)\,dx\right).$$

The weak form of the equation in this case $[\int_\Omega f(x)\,dx = 0]$ is therefore equivalent to

$$a(u, v) = (f, v) \qquad \text{for all} \qquad v \in V,$$

where

$$V = \left\{u \in H^1(\Omega) : \int_\Omega u(x)\,dx = 0\right\}.$$

It is was shown in Exercise 5.9 that in this space

$$|u| \leq C|\nabla u|,$$

and so we have

$$|a(u, u)| = |\nabla u|^2 \geq \frac{1}{2C}|u|^2 + \tfrac{1}{2}|\nabla u|^2 \geq k\|u\|_{H^1}^2.$$

We can therefore apply the Lax–Milgram lemma to deduce the existence of a weak solution of the Neumann problem.

6.6 Without the imposition of the condition $\int_Q u(x)\,dx = 0$ Laplace's equation on Q with periodic boundary conditions does not have a unique weak solution. In terms of the Lax–Milgram lemma this translates into the weak problem

$$a(u, v) = \int_Q \nabla u \cdot \nabla v \, dx = \langle f, v \rangle \qquad \text{with} \qquad f \in H^{-1}(Q),$$

where we seek $u \in L^2(Q)$. But then a is not coercive on $L^2(Q)$, since $a(c, c) = 0$ for any constant c.

6.7 First,

$$\begin{aligned} D_i^h(uv)(x) &= \frac{u(x + he_i)v(x + he_i) - u(x)v(x)}{h} \\ &= u(x)\left[\frac{v(x + he_i) - v(x)}{h}\right] + v(x + he_i)\left[\frac{u(x + he_i) - u(x)}{h}\right] \\ &= u(x)D_i^h v(x) + v(x + he_i)D_i^h u(x). \end{aligned}$$

Next, we write

$$\begin{aligned} &\int_\Omega \frac{u(x + he_i) - u(x)}{h} v(x) \, dx \\ &\quad = \int_\Omega \frac{u(x + he_i)}{h} v(x) \, dx - \int_\Omega \frac{u(x)}{h} v(x) \, dx \end{aligned}$$

and change variables in the first integral, putting $y = x + he_i$, to obtain

$$\begin{aligned} &\int_\Omega \frac{u(y)}{h} v(y - he_i) \, dy - \int_\Omega \frac{u(x)}{h} v(x) \, dx \\ &\quad = - \int_\Omega u(x) \frac{v(x - he_i) - v(x)}{-h} \, dx \\ &\quad = - \int_\Omega u(x) D_i^{-h} v(x) \, dx. \end{aligned}$$

Finally, both expressions are equal to

$$\frac{D_i u(x + he_j) - D_i u(x)}{h}.$$

6.8 The inverse of Φ is just the map $y \mapsto x$, given by

$$x_i = \begin{cases} y_i + z_i, & i = 1, \dots, m - 1, \\ y_m + \psi(y_1 + z_1, \dots, y_{m-1} + z_{m-1}), & i = m. \end{cases}$$

Therefore

$$\nabla\psi = \begin{pmatrix} 1 & 0 & 0 & \dots & D_1\psi \\ 0 & 1 & 0 & \dots & D_2\psi \\ 0 & 0 & 1 & \dots & D_3\psi \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

It follows immediately that $\det \nabla\Psi = 1$, as required.

6.9 The result of Lemma 3.26 shows that, for a general positive symmetric linear operator whose inverse is compact,

$$|A^s u| \leq C|A^l u|^{(k-s)/(k-l)}|A^k u|^{(s-l)/(k-l)}$$

for $0 \leq l < s < k$. $A = -\Delta$ on Ω' with Dirichlet boundary conditions certainly satisfies these conditions.

Taking $u \in H_0^k(\Omega')$, Proposition 6.19 shows that $u \in D(A^{j/2})$ for all $j = 0, 1, \dots, k$, and so

$$\|u\|_{H^j(\Omega')} \leq |A^{j/2}u| \leq C_j\|u\|_{H^j(\Omega')}.$$

Therefore we have

$$\|u\|_{H^s(\Omega')} \leq C\|u\|_{H^l(\Omega')}^{(k-s)/(k-l)}\|u\|_{H^k(\Omega')}^{(s-l)/(k-l)} \tag{S6.1}$$

for all such u.

Now take $u \in H^k(\Omega)$, and use Theorem 5.20 to extend u to a function $Eu \in H_0^k(\Omega')$ for some $\Omega' \supset \Omega$. Then (S6.1) holds for Eu, and since E is bounded from $H^j(\Omega)$ into $H_0^j(\Omega')$ for each $0 \leq j \leq k$, we have

$$\begin{aligned} \|u\|_{H^s(\Omega)} \leq \|Eu\|_{H^s(\Omega')} &\leq C\|Eu\|_{H^l(\Omega')}^{(k-s)/(k-l)}\|Eu\|_{H^k(\Omega')}^{(s-l)/(k-l)} \\ &\leq C\|u\|_{H^l(\Omega)}^{(k-s)/(k-l)}\|u\|_{H^k(\Omega)}^{(s-l)/(k-l)}, \end{aligned}$$

which is (6.32) for $u \in H^k(\Omega)$, as required.

Chapter 7

7.1 Define an element $I \in X^{**}$ by

$$\langle I, L\rangle = \int_0^T \langle L, f(t)\rangle\, dt \qquad \text{for all} \qquad L \in X^*. \tag{S7.1}$$

This map I is clearly linear, and it is bounded since

$$\begin{aligned}|\langle I, L\rangle| &\le \int_0^T \|L\|_{X^*}\|f(t)\|_X\,dt\\ &\le \left(\int_0^T \|f(t)\|_X\,dt\right)\|L\|_{X^*},\end{aligned}$$

and

$$\int_0^T \|f(t)\|_X\,dt < \infty$$

from (7.31). Since X is reflexive, it follows that there exists an element $y \in X$ such that

$$\langle I, L\rangle = \langle L, y\rangle \qquad \text{for all} \qquad L \in X^*.$$

Therefore, using (S7.1), we have (7.29).

That the integral is well defined follows from Lemma 4.4, which shows that if

$$\langle L, y_1\rangle = \langle L, y_2\rangle \qquad \text{for all} \qquad L \in X^*$$

then $y_1 = y_2$.

7.2 Corollary 4.5 shows that there exists an element $L \in X^*$ such that $\|L\|_{\mathrm{op}} = 1$ and $Ly = \|y\|_X$. Then, using (7.29), we have

$$\begin{aligned}\left\|\int_0^T f(t)\,dt\right\|_X &\le \int_0^T |\langle L, f(t)\rangle|\,dt\\ &\le \int_0^T \|f(t)\|_X\,dt,\end{aligned}$$

as required.

7.3 An element v of $L^p(0, T; V)$ is the limit in the L^p norm of a sequence of functions v_n in $C^0([0, T]; V)$. Since such functions are uniformly continuous on $[0, T]$, given $\epsilon > 0$ we can find an integer N such that $\delta = T/N$ satisfies

$$|t - s| < \delta \qquad \Rightarrow \qquad \|v_n(t) - v_n(s)\|_V \le \epsilon/T^{1/p}.$$

We can approximate v_n to within ϵ in $L^p(0, T; V)$ by

$$\sum_{j=1}^N v_n(j\delta)\chi[(j\delta, (j+1)\delta)],$$

an expression of the form (7.32). It follows that such elements are dense in $L^p(0, T; V)$. Since $C^1([0, T])$ is dense in $L^p(0, T)$ we could also use elements of the form of (7.32) with $\alpha_j \in C^1([0, T])$; similarly, $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, so we could take $v_j \in C_c^\infty(\Omega)$.

7.4 Taking the inner product of (7.33) with $A^k u_n$ yields

$$\frac{1}{2}\frac{d}{dt}|A^{k/2}u_n|^2 + \left|A^{\frac{k+1}{2}}u_n\right|^2 \le \left|A^{\frac{k-1}{2}}f\right|\left|A^{\frac{k+1}{2}}u_n\right|,$$

and so, using Young's inequality, we obtain

$$\frac{d}{dt}|A^{k/2}u_n|^2 + \left|A^{\frac{k+1}{2}}u_n\right|^2 \le \left|A^{\frac{k-1}{2}}f\right|^2,$$

which shows that

$$|A^{k/2}u_n(t)|^2 + \int_0^t \left|A^{\frac{k+1}{2}}u_n(s)\right|^2 ds \le |A^{k/2}u(0)|^2 + \left|A^{\frac{k-1}{2}}f\right|^2,$$

which yields (7.34), and then (7.35) follows from (7.33). Therefore, using Proposition 6.18, we get

$$u_n \in L^\infty(0, T; H^k) \cap L^2(0, T; H^{k+1})$$

and

$$du_n/dt \in L^2(0, T; H^{k-1}).$$

Extracting a subsequence shows that the solution u satisfies

$$u \in L^2(0, T; H^{k+1}) \qquad \text{and} \qquad du/dt \in L^2(0, T; H^{k-1}).$$

It follows from Corollary 7.3 that $u \in C^0([0, T]; H^k)$.

7.5 Since the $\{w_j\}$ are orthogonal in H_0^1 and orthonormal in L^2 the equation for u_n becomes

$$\lambda_j u_{nj} = f_j,$$

where $\lambda_j \equiv \|w_j\|^2$ and $f_j = (f, w_j)$. It follows that

$$u_{nj} = f_j/\lambda_j,$$

independent of n. In particular we have

$$u_n = \sum_{j=1}^n \frac{f_j}{\lambda_j} w_j,$$

and so, for $m > n$,

$$\|u_m - u_n\|^2 \leq \sum_{j=n+1}^{m} \frac{f_j^2}{\lambda_j}.$$

Since we have the Poincaré inequality we must have $\lambda_j \geq \lambda$, for some λ, and so

$$\|u_m - u_n\|^2 \leq \frac{1}{\lambda} \sum_{j=n+1}^{m} f_j^2.$$

Since $f \in L^2(\Omega)$ it follows that u_n converges in $H_0^1(\Omega)$ to $u = \sum_{j=1}^{\infty} f_j w_j / \lambda_j$.

Now, we know that

$$((u_n, v)) = (P_n f, v) \qquad \text{for all} \qquad v \in P_n H_0^1(\Omega).$$

Since

$$((u_n, v)) = ((u_n, P_n v)) \qquad \text{and} \qquad (P_n f, v) = (P_n f, P_n v)$$

for all $v \in H_0^1(\Omega)$, we in fact have

$$((u_n, v)) = (P_n f, v) \qquad \text{for all} \qquad v \in H_0^1(\Omega).$$

Since $u_n \to u$ in $H_0^1(\Omega)$ we know that

$$((u_n, v)) \to ((u, v)),$$

and since $P_n f \to f$ in $L^2(\Omega)$ we must have

$$((u, v)) = (f, v) \qquad \text{for all} \qquad v \in H_0^1(\Omega),$$

and u is a weak solution of (7.36) as required.

Chapter 8

8.1 We show in general that if $Z = X \cap Y$, with norm

$$\|u\|_Z = \|u\|_X + \|u\|_Y,$$

then $Z^* = X^* + Y^*$. First, it is clear that if $f = f_1 + f_2$, with $f_1 \in X^*$ and $f_2 \in Y^*$, then for $u \in X \cap Y$

$$\begin{aligned}|\langle f_1 + f_2, u\rangle| &\le |\langle f_1, u\rangle| + |\langle f_2, u\rangle| \\ &\le \|f_1\|_{X^*}\|u\|_X + \|f_2\|_{Y^*}\|u\|_Y \\ &\le (\|f_1\|_{X^*} + \|f_2\|_{Y^*})\|u\|_{X\cap Y}.\end{aligned}$$

Thus $X^* + Y^* \subset (X \cap Y)^*$. Now, if $f \in (X \cap Y)^*$ then, since it is a linear functional on a linear subspace of X, application of the Hahn–Banach theorem (Theorem 4.3) tells us it has an extension f_1 that is a linear functional on the whole of X (we could use Y rather than X here if we wished). Thus $(X \cap Y)^* \subset X^* \subset X^* + Y^*$, and so we have the required equality.

8.2 Follow the argument of Theorem 7.2, except approximate u by a sequence $u_n \in C^1([0, T]; H^1)$ such that

$$u_n \to \qquad \text{in} \qquad L^2(0, T; H^1) \cap L^p(\Omega_T)$$

and

$$du_n/dt \to du/dt \qquad \text{in} \qquad L^2(0, T; H^{-1}) + L^q(\Omega_T).$$

We will denote by $X(t_1, t_2)$ the space

$$L^2(t_1, t_2; H^1) \cap L^p(\Omega \times (t_1, t_2)),$$

and by $X^*(t_1, t_2)$ the space

$$L^2(t_1, t_2; H^{-1}) + L^q(\Omega \times (t_1, t_2)).$$

We now estimate

$$\begin{aligned}\int_\Omega |u_n(t)|^2\,dx &= \frac{1}{T}\int_\Omega\int_0^T |u_n(t)|^2\,dt\,dx + 2\int_\Omega\int_{t^*}^t \dot u_n(s)u_n(s)\,ds \\ &\le \frac{1}{T}\int_\Omega\int_0^T |u_n(t)|^2\,dt\,dx + 2\|\dot u_n\|_{X^*(t^*,t)}\|u_n\|_{X(t^*,t)} \\ &\le \frac{1}{T}\int_\Omega\int_0^T |u_n(t)|^2\,dt\,dx + 2\|\dot u_n\|_{X^*(0,T)}\|u_n\|_{X(0,T)},\end{aligned}$$

showing once again that u_n is also a Cauchy sequence in $C^0([0, T]; L^2)$ and hence that $u \in C^0([0, T]; L^2)$ as claimed.

8.3 Integrating by parts gives

$$-\int_\Omega \sum_j f(u_n)\frac{\partial^2 u_n}{\partial x_j^2}\,dx$$
$$= \int_\Omega \sum_j f'(u_n)\left|\frac{\partial u_n}{\partial x_j}\right|^2 dx + \int_{\partial\Omega} f(u_n)\nabla u_n\cdot n\,dS.$$

We can estimate the extra term by

$$\begin{aligned}\int_{\partial\Omega} f(u_n)\nabla u_n\cdot n\,dS &\le |f(0)|\int_{\partial\Omega}|\nabla u_n|\,dS\\ &\le |f(0)||\partial\Omega|^{1/2}\|\nabla u_n\|_{L^2(\partial\Omega)}\\ &\le C\|u_n\|_{H^1(\partial\Omega)}\\ &\le C\|u_n\|_{H^2(\Omega)},\end{aligned}$$

using the trace theorem (Theorem 5.35). Since we have

$$\|u\|_{H^2(\Omega)} \le C|Au|$$

from Theorem 6.16, we can write

$$\tfrac{1}{2}\frac{d}{dt}\|u_n\|^2 + |Au_n|^2 \le l\|u_n\|^2 + C|Au_n|.$$

Using Young's inequality on the last term and rearranging finally gives

$$\frac{d}{dt}\|u_n\|^2 + |Au_n|^2 \le 2l\|u_n\|^2 + C,$$

which integrates to give the bound

$$\|u_n(T)\|^2 + \int_0^T |Au_n(s)|^2\,ds \le 2l\int_0^T \|u_n(t)\|^2\,dt + \|u_0\|^2 + CT.$$

Thus u_n is uniformly bounded in $L^2(0,T;D(A))$ [and $L^\infty(0,T;V)$], where (8.19) is used as before to guarantee that $u_n \in L^2(0,T;V)$.

8.4 In this case we can follow the proof of Proposition 8.6 until the line

$$|F(u) - F(v)|^2 \le C|u-v|_{L^{2p}}^2\bigl(1 + |u|_{L^{2q\gamma}}^2 + |v|_{L^{2q\gamma}}^2\bigr).$$

We now have to be more careful with our use of the Sobolev embedding theorem, since the highest we can go is $H^1 \subset L^6$. We therefore need

$2p \le 6$ and $2q\gamma \le 6$, where (p,q) are conjugate.

The first conditions forces us to take $p \le 3$, and hence we must have

$q \geq 3/2$, which shows that the largest possible value for γ is 2, as claimed. Provided that $\gamma \leq 2$ we can write

$$|F(u) - F(v)|^2 \leq C\|u - v\|_{H^1}^2(1 + \|u\|_{H^1} + \|v\|_{H^1}),$$

as in Proposition 8.6.

8.5 Since $u, v \in L^2(0, T; D(A)$ we can use Corollary 7.3 to take the inner product of

$$\frac{dw}{dt} + Aw = F(u) - F(v)$$

with Aw to obtain, using (8.31),

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}\|w\|^2 + |Aw|^2 &= (F(u) - F(v), Aw)\\ &\leq C(1 + |Au| + |Av|)^{1/2}\|w\|^{1/2}|Aw|^{3/2}.\end{aligned}$$

We now use Young's inequality to split the right-hand side,

$$\tfrac{1}{2}\frac{d}{dt}\|w\|^2 + |Aw|^2 \leq \frac{3}{4}|Aw|^2 + C(1 + |Au| + |Av|)^2\|w\|^2,$$

and so

$$\tfrac{1}{2}\frac{d}{dt}\|w\|^2 \leq C(1 + |Au| + |Av|)^2\|w\|^2.$$

This yields

$$\|w(t)\|^2 \leq \|w(0)\|^2 \exp\left(\int_0^t C(1 + |Au(s)|^2 + |Av(s)|^2)\, ds\right),$$

which gives continuous dependence on initial conditions since we know that both u and v are elements of $L^2(0, T; D(A))$.

8.6 Setting $g(s) = e^{-A(t-s)}u(s)$, we have

$$\begin{aligned}\frac{\partial g}{\partial s} &= Ae^{-A(t-s)}u(s) + e^{-A(t-s)}\frac{du}{ds}\\ &= Ae^{-A(t-s)}u(s) + e^{-A(t-s)}[-Au + f(u(s))]\\ &= e^{-A(t-s)}f(u(s)),\end{aligned}$$

so that integrating with respect to s between 0 and t gives

$$g(t) - g(0) = \int_0^t e^{-A(t-s)}f(u(s))\, ds.$$

This rearranges to give (8.33).

Chapter 9

9.1 Taking the divergence of the governing equation yields

$$\Delta u = \nabla \cdot f,$$

since all the other terms are divergence free. A solution of this equation in the periodic case when $f \in \dot{L}^2(Q)$ has been obtained as Equation (9.10). Note that if $f \in H$ then this implies that $p = 0$ (or, equivalently, a constant).

9.2 We have

$$\begin{aligned} |u|_{L^4}^4 = \int_Q |u|^4 \, dx &\le \left(\int_Q |u|^6 \right)^{1/2} \left(\int_Q |u|^2 \right)^{1/2} \\ &= \|u\|_{L^6}^3 |u| \\ &\le k\|u\|^3 |u|, \end{aligned}$$

since $H^1(Q) \subset L^6(Q)$ (see Theorem 5.31).

9.3 Applying the Cauchy–Schwarz inequality first in the variable j and then in the variable i, we get

$$\begin{aligned} \left| \sum_{i,j=1}^m a_i b_{i,j} c_j \right| &\le \left(\sum_{i,j=1}^m |a_i b_{i,j}|^2 \right)^{1/2} \left(\sum_{j=1}^m |c_j|^2 \right)^{1/2} \\ &= \left(\sum_{i=1}^m \left| a_i \left(\sum_{j=1}^m b_{i,j} \right) \right| \right) \left(\sum_{j=1}^m |c_j|^2 \right)^{1/2} \\ &\le \left(\sum_{i=1}^m |a_i|^2 \right)^{1/2} \left(\sum_{i,j=1}^m |b_{i,j}|^2 \right)^{1/2} \left(\sum_{j=1}^m |c_j|^2 \right)^{1/2}, \end{aligned}$$

as claimed.

9.4 If $m = 2$, we have

$$|b(u, v, w)| \le k|u|^{1/2}\|u\|^{1/2}|v|^{1/2}\|v\|^{1/2}\|w\|$$

[using $b(u, v, w) = -b(u, w, v)$], so that

$$\langle B(u, u), w\rangle \le k|u|\|u\|\|w\|,$$

and therefore

$$\|B(u, u)\|_{V^*} \le k|u|\|u\|.$$

If $m = 3$, we have

$$|b(u, v, w)| \leq k|u|^{1/4}\|u\|^{3/4}|v|^{1/4}\|v\|^{3/4}\|w\|$$

[using $b(u, v, w) = -b(u, w, v)$ again], and so

$$\langle B(u, u), w\rangle \leq k|u|^{1/2}\|u\|^{3/2}\|w\|,$$

giving

$$\|B(u, u)\|_{V^*} \leq k|u|^{1/2}\|u\|^{3/2}.$$

9.5 Take $(p, q) = (2, 2)$ if $m = 2$ and $(p, q) = (4/3, 4)$ if $m = 3$. We know that $B_n \overset{*}{\rightharpoonup} B$ in $L^p(0, T; V^*)$, where $B_n = B(u_n, u_n)$ and $B = B(u, u)$. We need to show that $P_n B_n \overset{*}{\rightharpoonup} B$ in the same sense. For $\psi \in L^q(0, T; V)$ we have

$$\int_0^T \langle P_n B_n(t) - B, \psi\rangle \, dt = \int_0^T \langle P_n B_n - B_n, \psi\rangle \, dt + \int_0^T \langle B_n - B, \psi\rangle \, dt.$$

The second term converges since $B_n \overset{*}{\rightharpoonup} B$, so we have to treat only the first term. We rewrite this as

$$\int_0^T \langle B_n(t), Q_n\psi\rangle \, dt.$$

Since functions of the form

$$\psi = \sum_{j=1}^{k} \psi_j \alpha_j(t), \qquad \psi_j \in V,\ \alpha_j \in C^1([0, T], \mathbb{R}) \tag{S9.1}$$

are dense in $L^q(0, T; V)$ (see Exercise 7.3) we can consider

$$\int_0^T \left\langle B_n, \sum_{j=1}^{k} Q_n \psi_j \right\rangle \alpha_j(t) \, dt.$$

Since B_n is uniformly bounded in $L^p(0, T; V^*)$ when $m = 2$, we can use the fact that $Q_n\psi_j \to \psi_j$ in V to show the required convergence for all ψ of the form (S9.1). The density of such ψ in $L^q(0, T; V)$ then gives the full result.

9.6 If $u \in L^4(0,T;V)$ then we can estimate $b(w,u,w)$ in (9.41) differently, writing

$$\tfrac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 \le k|w|^{1/2}\|w\|^{3/2}\|u\| \\ \le \frac{\nu}{2}\|w\|^2 + \frac{c}{\nu^3}|w|^2\|u\|^4,$$

which becomes, dropping the terms in $\|w\|^2$,

$$\frac{d}{dt}|w|^2 \le C|w|^2\|u\|^4.$$

Integrating gives

$$|w(t)|^2 \le |w(0)|^2 \exp\left(\int_0^t \|u(s)\|^4\, ds\right),$$

which implies uniqueness provided that $u \in L^4(0,T;V)$.

Chapter 10

10.1 If not, then there exist an $\epsilon > 0$ and sequences $\delta_n \to 0$, $x_n \in K$, $y_n \in H$, such that

$$|x_n - y_n| \le \delta_n \qquad \text{and} \qquad |f(x_n) - f(y_n)| > \epsilon.$$

Since K is compact there is a subsequence of the $\{x_n\}$ (relabel this x_n) such that $x_n \to x^* \in X$. Now,

$$|x^* - y_n| \le |x^* - x_n| + |x_n - y_n| \to 0 \qquad \text{as} \qquad n \to \infty, \tag{S10.1}$$

and

$$|f(x^*) - f(y_n)| \ge |f(x_n) - f(y_n)| - |f(x_n) - f(x^*)| \ge \epsilon/2 \tag{S10.2}$$

if n is sufficiently large, since f is continuous at x^*. But then (S10.1) and (S10.2) say precisely that f is not continuous at x^*, which is a contradiction.

10.2 The set in (10.23) is bounded since

$$\bigcup_{t \ge t_0(B)} S(t)B, \tag{S10.3}$$

with $t_0(B)$ from Definition 10.2, is a subset of B, and

$$\overline{\bigcup_{0\le t\le t_0(B)} S(t)B} \tag{S10.4}$$

is bounded since B is bounded and $S(t)$ is continuous. Similarly, if B is compact then (S10.3) is a closed subset of B, and (S10.4) is the continuous image of the compact set $B \times [0, t_0(B)]$: both parts are compact, and therefore so is (10.23). That (10.23) is positively invariant is clear by definition.

10.3 In this example $\omega(0) = 0$ and $\omega(x) = \{|x| = 1\}$ if $x \neq 0$. So

$$\Lambda(B) = \{(0,0)\} \cup \{|x| = 1\}$$

(which is clearly not connected). Since $\omega(x) = (1,0)$ for all x with $|x| = 1$,

$$\Lambda[\Lambda(B)] = \{(0,0), (1,0)\},$$

so that $\Lambda[\Lambda(B)] \neq \Lambda(B)$ as claimed.

10.4 We show that, for a bounded set X,

$$\omega_1(X) = \{y : S(t_n)x_n \to y\},$$

where $t_n \to \infty$ and $x_n \in X$, is equal to

$$\omega_2(X) = \bigcap_{t\ge 0} \overline{\bigcup_{s\ge t} S(s)X}.$$

If $y \in \omega_1(X)$ then clearly

$$y \in \overline{\bigcup_{s\ge t} S(s)X}$$

for all $t \ge 0$ and hence in $y \in \omega_2(X)$. So $\omega_1(X) \subset \omega_2(X)$.

Conversely, if $y \in \omega_2(X)$ then for any $t \ge 0$

$$y \in \overline{\bigcup_{s\ge t} S(s)X},$$

and so there are sequences $\{\tau_m^{(t)}\}$, with $\tau_m^{(t)} \ge t$, and $\{x_m^{(t)}\} \in X$ with $S(\tau_m^{(t)})x_m^{(t)} \to y$. Now consider $t = 1, 2, \ldots$ and pick t_n from $\tau_m^{(n)}$ and

x_n from $x_m^{(n)}$ such that

$$\left|S\left(\tau_m^{(n)}\right)x_m^{(n)} - y\right| \le 1/n.$$

Then $S(t_n)x_n \to y$ with $t_n \to \infty$, since $t_n \ge n$, showing that $y \in \omega_1(X)$. This gives $\omega_2(X) \subset \omega_1(X)$, and so $\omega_1(X) = \omega_2(X)$.

10.5 If $y \in S(t)B$ for all $t \ge 0$ then for any t_n there is an $x_n \in B$ with $y = S(t_n)x_n$, so clearly $y \in \omega_2(B)$ (as defined in the previous solution). Conversely, if $y \in \omega_2(B)$ then we must have

$$y \in \overline{\bigcup_{s \ge t} S(s)B}.$$

Now, if $\tau \ge t_0(B)$, then

$$S(t)B \supset S(t+\tau)B,$$

and so then

$$S(t)B \supset \bigcup_{\tau \ge t_0(B)} S(t+\tau)B.$$

Since $S(t)B$ is closed

$$S(t)B \supset \overline{\bigcup_{\tau \ge t_0(B)} S(t+\tau)B} \ni y,$$

that is, $y \in S(t)B$ for all $t \ge 0$.

Clearly we have

$$\bigcap_{t \ge 0} S(t)B \subset \bigcap_{n \in \mathbb{Z}^+} S(nT)B.$$

If $u \in S(nT)B$ for all $n \in \mathbb{Z}^+$ then in particular $u \in S(n_0 T)B$, provided that n_0 is large enough that $n_0 T \ge t_0(B)$, where

$$S(t)B \subset B \qquad \text{for all} \qquad t \ge t_0(B).$$

Since $u \in S(nT)B$ we have $u = S(nT)y$ with $y \in B$, and it follows that for all $t \ge 0$

$$S(t)u = S(t+nT)y = S(\tau)y \in B,$$

since $\tau \ge t_0(B)$. Therefore $u \in S(t)B$ for all $t \ge n_0 T$. It follows that $u \in \omega_2(B)$, giving the required equality.

10.6 $y \in \omega(Y)$ if $S(t_n)y_n \to y$ with $t_n \to \infty$ and $y_n \in Y$. Then $y_n \in X$ also and so $\omega(X) \supset \omega(Y)$. If Y absorbs X in a time t_0 (assuming X to be bounded) and if $S(t_n)x_n \to x$, then

$$S(t_n - t_0)[S(t_0)x_n] \to x,$$

and since $t_n - t_0 \to \infty$ and $S(t_0)x_n \in Y$, $\omega(X) \subset \omega(Y)$, so then $\omega(X) = \omega(Y)$.

10.7 First, the set

$$\overline{\bigcup_{i=j}^{\infty} K_i} \tag{S10.5}$$

is clearly closed, and since all sets K_i lie within $1/j$ of K_j if $i \geq j$ it is also bounded, and hence compact. It follows that K_∞, the intersection of a decreasing sequence of compact sets, is itself compact.

Now, it is clear by a similar argument that

$$\text{dist}(K_\infty, K_j) \leq j^{-1}.$$

Conversely, if $u \in K_j$ then $\text{dist}(u, K_i) \leq j^{-1}$ for all $i \geq j$. So certainly

$$\text{dist}\left(u, \bigcup_{i=j}^{\infty} K_i\right) \leq j^{-1}.$$

In particular, there exist points $u_i \in K_i$, $i \geq j$, such that

$$|u_i - u| \leq j^{-1}.$$

Since each u_i is contained in the compact set (S10.5) (with $j = 1$) then there exists a subsequence of the u_i that converges to some u^*. It follows that $u^* \in K_\infty$, and by construction $|u - u^*| \leq j^{-1}$. Therefore

$$\text{dist}(K_j, K_\infty) \leq j^{-1},$$

and so

$$\text{dist}_{\mathcal{H}}(K_j, K_\infty) \leq j^{-1} :$$

K_j converges to K_∞ in the Hausdorff metric.

10.8 To show that the inverse is continuous, suppose not. Then there exist an $\epsilon > 0$ and a sequence $\{x_n\} \in f(X)$ with $x_n \to y \in f(X)$ but $|f^{-1}(x_n) - f^{-1}(y)| \geq \epsilon$. However, $f^{-1}(x_n) \in X$, and since X is compact there exists a subsequence x_{n_j} such that $f^{-1}(x_{n_j}) \to z$. Since f is continuous, it follows that $x_{n_j} \to f(z)$. Since f is injective, it follows from $f(z) = y$ that $z = f^{-1}(y)$, which is a contradiction. So f^{-1} is continuous on $f(X)$.

10.9 Proposition 10.14 says that, given ϵ_1 and $T > 0$, there exists a time τ_1 such that, for all $t \geq \tau_1$,

$$\text{dist}(u(t), \mathcal{A}) \leq \delta(\epsilon_1, T).$$

So we can track the trajectory $u(t)$ within a distance ϵ_1 for a time T starting at any time $t \geq \tau_1$.

We can replace T with $2T$ and apply the same argument for $\epsilon_2 = \epsilon_1/2$, that is, there exists a time τ_2 such that, for all $t \geq \tau_2$,

$$\text{dist}(u(t), \mathcal{A}) \leq \delta(\epsilon_2, 2T),$$

and then the trajectory $u(t)$ can be tracked for a time $2T$ starting at any time $t \geq \tau_2$.

Thus $u(t)$ can be followed from τ_1 to τ_2 by a distance ϵ_1 with a finite number of trajectories on $\mathcal{A}$ of time length T, and when we reach τ_2, we can start to track $u(t)$ within a distance ϵ_2 with trajectories on $\mathcal{A}$ of time length $2T$, until we reach a τ_3 after which we can track within a distance ϵ_3 for a time length $3T$, etc.

The "jumps" are bounded by $\epsilon_k + \epsilon_{k+1}$, since

$$\begin{aligned} &|v_{k+1} - S(t_{k+1} - t_k)v_k| \\ &\quad \leq |v_{k+1} - u(t_{k+1})| + |u(t_k + (t_{k+1} - t_k)) - S(t_{k+1} - t_k)v_k| \\ &\quad \leq \epsilon_{k+1} + \epsilon_k. \end{aligned}$$

10.10 Take $\epsilon > 0$. Then there is a $T > 0$ such that

$$\text{dist}(S(t)B_1, \mathcal{A}) + \text{dist}(S(t)B_2, \mathcal{A}) < \epsilon \qquad \text{for all} \qquad t \geq T.$$

Also, by the uniform continuity of the semigroup, there is a $\delta > 0$ such that

$$\text{dist}(S(t)B_1, S(t)B_2) \leq \epsilon \qquad \text{for all} \qquad t \in [0, T]$$

provided that $\text{dist}(B_1, B_2) \leq \delta$. The argument is symmetric, which gives the result.

Chapter 11

11.1 Using Young's inequality on (11.30) we can deduce that

$$|u|^2 \le \frac{p}{2}\int_\Omega |u|^p\,dx + \frac{p}{p-2}|\Omega|.$$

So we can write (11.6) as

$$\tfrac{1}{2}\frac{d}{dt}|u|^2 + \|u\|^2 + \frac{2\alpha_2}{p}|u|^2 \le \left(\frac{2}{p-2} + k\right)|\Omega|.$$

Neglecting the $\|u\|^2$ term we can write

$$\tfrac{1}{2}\frac{d}{dt}|u|^2 + \frac{2\alpha_2}{p}|u|^2 \le \left(\frac{2}{p-2} + k\right)|\Omega|.$$

We can now apply the Gronwall inequality to deduce an asymptotic bound on $|u(t)|$, as in Proposition 11.1. (The expression for the bound will be a more complicated expression than before.)

11.2 Proceeding as advised, we obtain

$$\frac{d}{ds}\left(y(s)\exp\left(-\int_t^s g(\tau)\,d\tau\right)\right) \le h(s)\exp\left(-\int_t^s g(\tau)\,d\tau\right) \le h(s),$$

and integrating both sides between s and $t+r$ gives

$$\begin{aligned} y(t+r) &\le y(s)\exp\left(\int_s^{t+r} g(\tau)\,d\tau\right) \\ &\qquad + \left(\int_s^{t+r} h(\tau)\,d\tau\right)\exp\left(\int_s^{t+r} g(\tau)\,d\tau\right) \\ &\le (y(s)+a_2)\exp(a_1). \end{aligned}$$

Integrating both sides for $t \le s \le t+r$ gives the result as stated.

11.3 Taking the inner product of

$$\frac{du_n}{dt} + Au_n = P_n f(u_n)$$

with t^2Au_n we obtain

$$\left(\frac{du_n}{dt}, t^2Au_n\right) + t^2|Au_n|^2 = (P_n f(u_n), t^2Au_n),$$

which, using the methods leading to (8.27) for the right-hand side, becomes

$$\tfrac{1}{2}\frac{d}{dt}\|tu_n\|^2 - 2t\|u_n\|^2 + t^2|Au_n|^2 \leq lt^2\|u_n\|^2.$$

Integrating from 0 to T gives

$$\|Tu_n\|^2 + \int_0^T t^2|Au_n|^2\,dt \leq \int_0^T (2t + lt^2)\|u_n\|^2\,dt.$$

Since we already know that $u_n \in L^2(0, T; V)$, it follows that $u_n \in L^2(t, T; D(A))$ for any $t > 0$. Since $H^2(\Omega) \subset C^0(\overline{\Omega})$ if $m \leq 3$ we also have $P_n f(u_n) \in L^2(t, T; L^2)$, and so it follows that $du_n/dt \in L^2(t, T; H)$. Taking limits shows that the solution u satisfies

$$u \in L^2(t, T; D(A)) \qquad \text{and} \qquad du/dt \in L^2(t, T; L^2).$$

Application of Corollary 7.3 then makes the "formal" calculations at the beginning of Section 11.1.2 rigorous.

11.4 Observe that for $s < 0$ we have

$$f(s)|s| \geq \alpha_2|s|^p - k,$$

and so in particular

$$f(s) \geq 0 \qquad \text{for all} \qquad s < (k/\alpha_2)^{1/p}. \tag{S11.1}$$

Now set $M = (k/\alpha_2)^{1/p}$, multiply Equation (11.1) by $(u(x) + M)_-$, and integrate to obtain

$$\tfrac{1}{2}\frac{d}{dt}\int_\Omega (u(x) + M)_-^2 + \int_\Omega |\nabla(u + M)_-|^2 = \int_\Omega f(u)(u + M)_-\,dx \\ \leq 0,$$

using (S11.1). It follows, using the Poincaré inequality, that

$$\tfrac{1}{2}\frac{d}{dt}\int_\Omega (u(x) + M)_-^2\,dx \leq -C\int_\Omega (u(x) + M)_-^2\,dx,$$

and so as in the last part of the argument given in Theorem 11.6, we must have

$$\int_\Omega (u(x) + M)_-^2\,dx = 0$$

for all $u \in \mathcal{A}$.

Chapter 12

12.1 If u is smooth then $Au = -\Delta u$, and we have

$$
\begin{aligned}
b(u, u, A^2u) &= \sum_{i,j,k,l=1}^{2} \int_\Omega u_i (D_i u_j) D_k^2 D_l^2 u_j \, dx \\
&= \sum_{i,j,k,l=1}^{2} \int_\Omega D_k^2 (u_i (D_i u_j)) \big(D_l^2 u_j\big) \, dx \\
&= \sum_{i,j,k,l} \int_\Omega \big[\big(D_k^2 u_i\big)(D_i u_j) + 2(D_k u_i)(D_k D_i u_j) \\
&\quad + u_i \big(D_i D_k^2 u_j\big)\big] \big(D_l^2 u_j\big) \\
&= b(Au, u, Au) + 2 \sum_{i,j,k,l} (D_k u_i)(D_k D_i u_j) \big(D_l^2 u_j\big) \, dx \\
&\quad + b(u, Au, Au) \\
&= b(Au, u, Au) + 2 \sum_{k=1}^{2} b(D_k u, D_k u, Au),
\end{aligned}
$$

as claimed. The result follows for general u by taking limits.

To obtain inequality (12.23), use (9.26) to give

$$
\begin{aligned}
|b(u, u, A^2u)| &\le k|A^{3/2}u||Au|\|u\| + 2 \sum_{j=1}^{2} |b(D_j u, Au, D_j u)| \\
&\le k|A^{3/2}u||Au|\|u\| + 2k \sum_{j=1}^{2} |D_j u|\|D_j u\|\|Au\| \\
&\le k|A^{3/2}u||Au|\|u\| \\
&\quad + 2k \left(\sum_{j=1}^{2} |D_j u|^2 \right)^{1/2} \left(\sum_{j=1}^{2} \|D_j u\|^2 \right)^{1/2} \|Au\|.
\end{aligned}
$$

Since

$$
\|u\|^2 = a(u, u) = \langle Au, u \rangle = (A^{1/2}u, A^{1/2}u) = |A^{1/2}u|^2,
$$

this becomes

$$
\begin{aligned}
|b(u, u, A^2u)| &\le 3k|A^{3/2}u||Au|\|u\| \\
&\le \frac{\nu}{4}|A^{3/2}u|^2 + \frac{9k^2}{\nu}\|u\|^2|Au|^2,
\end{aligned}
$$

as required.

12.2 Take the inner product of

$$du/dt + \nu Au + B(u,u) = f$$

with A^2u to obtain

$$\tfrac{1}{2}\frac{d}{dt}|Au|^2 + \nu|A^{3/2}u|^2 = -b(u,u,A^2u) + (f, A^2u),$$

and use the estimate (12.23) from the previous exercise to write

$$\begin{aligned}
&\tfrac{1}{2}\frac{d}{dt}|Au|^2 + \nu|A^{3/2}u|^2 \\
&\leq |b(u,u,A^2u)| + \|f\||A^{3/2}u| \\
&\leq \frac{\nu}{4}|A^{3/2}u|^2 + \frac{C}{\nu}\|u\|^2|Au|^2 + \frac{\|f\|^2}{\nu} + \frac{\nu}{4}|A^{3/2}u|^2,
\end{aligned}$$

so that

$$\frac{d}{dt}|Au|^2 + \nu|A^{3/2}u|^2 \leq \frac{2\|f\|^2}{\nu} + \frac{C}{\nu}\|u\|^2|Au|^2.$$

Using a similar trick as we did for the absorbing set in V, we integrate this equation between s and t, with $t < s < t+1$, so that

$$|Au(t+1)|^2 \leq |Au(s)|^2 + \frac{2M}{\nu} + \frac{C}{\nu}\int_t^{t+1} \|u(s)\|^2|Au(s)|^2\,ds,$$

where we have used (12.24). Integrating again with respect to s between t and $t+1$ gives

$$|Au(t+1)|^2 \leq \int_t^{t+1} |Au(s)|^2\,ds + \frac{2M}{\nu} + \frac{C}{\nu}\int_t^{t+1} \|u(s)\|^2|Au(s)|^2\,ds. \tag{S12.1}$$

Now, if $t \geq t_1(|u_0|)$ then we know that

$$\|u(s)\| \leq \rho_V \qquad \text{and} \qquad \int_t^{t+1} |Au(s)|^2\,ds \leq I_A,$$

and so if it follows that then

$$|Au(t+1)|^2 \leq \rho_A \equiv I_A + \frac{2M}{\nu} + \frac{C}{\nu}\rho_V^2 I_A,$$

an absorbing set in $D(A)$.

12.3 Suppose that $u_n \in V$ with $\|u_n\| \leq k$ and that $u_n \to u$ in H. Then there exists a subsequence u_{n_j} such that $u_{n_j} \rightharpoonup v$ in V, so that $\|v\| \leq k$. Since $V \subset\subset H$, it follows that $u_{n_j} \to v$ in H, and so in particular we must have $u = v$, which implies that $\|u\| \leq k$.

12.4 If $u \in D(A)$ with

$$u = \sum_{k \in \mathbb{Z}^2} u_k e^{2\pi i k \cdot x / L}$$

then we can estimate $\|u\|_\infty$ by

$$\|u\|_\infty \leq \sum_{k \in \mathbb{Z}^2} |u_k|.$$

Split the sum into two parts,

$$\|u\|_\infty \leq \sum_{|k| \leq \kappa} |u_k| + \sum_{|k| > \kappa} |u_k|.$$

We now use the Cauchy–Schwarz inequality on each piece,

$$\begin{aligned}
\|u\|_\infty &\leq \sum_{|k| \leq \kappa} (|u_k| \times 1) + \sum_{|k| > \kappa} (|u_k||k|^2 \times |k|^{-2}) \\
&\leq \left(\sum_{|k| \leq \kappa} |u_k|^2 \right)^{1/2} \left(\sum_{|k| \leq \kappa} 1 \right)^{1/2} \\
&\quad + \left(\sum_{|k| > \kappa} |u_k|^2 |k|^4 \right)^{1/2} \left(\sum_{|k| > \kappa} |k|^{-4} \right)^{1/2}.
\end{aligned}$$

Since

$$\sum_{|k| \leq \kappa} 1 \leq C\kappa^2 \qquad \text{and} \qquad \sum_{|k| > \kappa} |k|^4 \leq C\kappa^{-2},$$

this becomes

$$\|u\|_\infty \leq C(\kappa |u| + \kappa^{-1} |Au|).$$

To make both terms on the right-hand side the same, we choose $\kappa = |Au|^{1/2} |u|^{-1/2}$, obtaining

$$\|u\|_\infty \leq C |u|^{1/2} |Au|^{1/2}.$$

12.5 We have already derived in (12.20) the inequality

$$\frac{d}{dt} \|u\|^2 + \nu |Au|^2 \leq \frac{2|f|^2}{\nu} + C\|u\|^6,$$

and since we have a uniform bound on $\|u\|$ for t large enough, we obtain a uniform bound on the integral of $|Au(s)|^2$,

$$\int_{t_0}^{t_0+1} |Au(s)|^2 \, ds \leq C_1. \tag{S12.2}$$

Following the analysis in Proposition 12.4, we estimate

$$|u_t| \le \nu|Au| + |B(u,u)| + |f|,$$

and using (12.25) this becomes

$$|u_t| \le \nu|Au| + k\|u\|^{3/2}|Au|^{1/2} + |f|.$$

An application of Young's inequality yields

$$|u_t| \le c|Au| + C\|u\|^2 + |f|,$$

and so for t large enough,

$$|u_t| \le c|Au| + C\rho_V^2 + |f|.$$

The bound in (S12.2) therefore implies a bound on $\int |u_t|^2$,

$$\int_{t_0}^{t_0+1} |u_t(s)|^2\,ds \le C_2. \tag{S12.3}$$

Now differentiate

$$u_t + \nu Au + B(u,u) = f$$

with respect to t to obtain

$$u_{tt} + \nu Au_t + B(u_t,u) + B(u,u_t) = 0$$

and take the inner product with u_t so that

$$\begin{aligned} \tfrac{1}{2}\frac{d}{dt}|u_t|^2 + \nu\|u_t\|^2 &\le |b(u_t,u,u_t)| \\ &\le k\|u\||u_t|^{1/2}\|u_t\|^{3/2} \\ &\le \frac{3\nu}{4}\|u_t\|^2 + \frac{k^4\|u\|^4|u_t|^2}{4\nu^3}. \end{aligned}$$

Using once again the asymptotic bound on $\|u\|$, we have for $t \ge t_0$ that

$$\frac{d}{dt}|u_t|^2 \le C_3|u_t|^2.$$

We use the usual trick, integrating between s and $t+1$, with $t < s < t+1$,

$$|u_t(t+1)|^2 \le |u_t(s)|^2 + C_4\int_t^{t+1} |u_t(s)|^2\,ds,$$

and then between t and $t+1$ (with respect to s) so that

$$\begin{aligned}|u_t(t+1)|^2 &\leq (1+C_4)\int_t^{t+1} |u_t|^2\,ds \\ &\leq (1+C_4)C_3, \qquad \text{(S12.4)}\end{aligned}$$

by (S12.3).

To end, we show that $|u_t|$ bounds $|Au|$. From the equation we have

$$\nu|Au| \leq |u_t| + |B(u,u)| + |f|,$$

or with (12.25)

$$\nu|Au| \leq |u_t| + k|Au|^{1/2}\|u\|^{3/2} + |f|,$$

and so after using Young's inequality and rearranging we have

$$|Au| \leq C(|u_t| + \|u\|^3 + |f|).$$

Together with (S12.4) we obtain

$$|Au(t)| \leq \rho_D$$

for all $t \geq 1 + t_0(\|u_0\|)$. So we have an absorbing set in $D(A)$ and hence a global attractor for the 3D equations.

Chapter 13

13.1 Let $G(X,\epsilon)$ be the number of boxes in a fixed cubic lattice, with sides ϵ, that are necessary to cover X. Since each cube with side ϵ sits inside a ball of radius ϵ, $N(X,\epsilon) \leq G(X,\epsilon)$, and so

$$d_f(X) \leq d_{\text{box}}(X).$$

Also, since any ball with side ϵ is contained within at most 2^m different boxes in the grid, we have $G(X,\epsilon) \leq 2^m N(X,\epsilon)$. Therefore

$$\begin{aligned}d_{\text{box}}(X) &= \limsup_{\epsilon\to 0} \frac{\log G(X,\epsilon)}{-\log\epsilon} \\ &\leq \limsup_{\epsilon\to 0} \frac{m\log 2 + \log N(X,\epsilon)}{-\log\epsilon} \\ &= \limsup_{\epsilon\to 0} \frac{\log N(X,\epsilon)}{-\log\epsilon} \\ &= d_f(X),\end{aligned}$$

giving equality between box-counting dimension and fractal dimension in $\mathbb{R}^m$.

13.2 If $\epsilon_{n+1} \le \epsilon < \epsilon_n$ then we have

$$\begin{aligned}\frac{\log N(X,\epsilon)}{-\log\epsilon} &\le \frac{\log N(X,\epsilon_{n+1})}{-\log\epsilon_n} \\ &\le \frac{\log N(X,\epsilon_{n+1})}{-\log\epsilon_{n+1} + \log(\epsilon_{n+1}/\epsilon_n)} \\ &\le \frac{\log N(X,\epsilon_{n+1})}{-\log\epsilon_{n+1} + \log\alpha},\end{aligned}$$

and so

$$\limsup_{\epsilon\to 0} \frac{\log N(X,\epsilon)}{-\log\epsilon} \le \limsup_{n\to\infty} \frac{\log N(X,\epsilon_n)}{-\log\epsilon_n}.$$

That this inequality holds in the opposite sense is straightforward, and hence we obtain the desired equality.

13.3 The sequence $\epsilon_m = (\sqrt{2}\log m)^{-1}$, $m \ge 2$, satisfies

$$\frac{\epsilon_{m+1}}{\epsilon_m} = \frac{\log m}{\log(m+1)} \ge \frac{\log 2}{\log 3},$$

and so we can use the result of the previous exercise. Note that we have

$$\left|\frac{e_n}{\log n} - \frac{e_k}{\log k}\right|^2 = \frac{1}{(\log n)^2} + \frac{1}{(\log k)^2} \le \frac{2}{(\log n)^2}$$

for $n > k$, and so the first $m-1$ elements from $H_{\log}$ will belong to distinct balls of radius ϵ_m. It follows that

$$N(H_{\log}) \ge m - 1,$$

and so

$$\begin{aligned} d_f(H_{\log}) &\ge \limsup_{m\to\infty} \frac{\log N(H_{\log},\epsilon_m)}{\log\epsilon_m} \\ &\ge \limsup_{m\to\infty} \frac{\log(m-1)}{\log(\sqrt{2}\log m)} = \infty,\end{aligned}$$

which implies that $d_f(H_{\log}) = \infty$, as claimed.

13.4 At the jth stage of construction the middle-α set C_α consists of 2^j intervals of length β^j, where $\beta = (1-\alpha)/2$. It follows that

$$N(C_\alpha, \beta^j) = 2^j.$$

Therefore, using the result of Exercise 13.2 we can calculate

$$d_f(C_\alpha) = \limsup_{j\to\infty} \frac{\log 2^j}{\log \beta^j} = \frac{\log 2}{\log \beta}.$$

13.5 Clearly

$$\mu\left(\bigcup_{k=1}^{\infty} X_k, d, \epsilon\right) \le \sum_{k=1}^{\infty} \mu(X_k, d, \epsilon).$$

Since $\mu(X_k, d, \epsilon)$ is nondecreasing in ϵ we have

$$\mu(X_k, d, \epsilon) \le \mathcal{H}^d(X_k)$$

for each k, and so for every $\epsilon > 0$ we have

$$\mu\left(\bigcup_{k=1}^{\infty} X_k, d, \epsilon\right) \le \sum_{k=1}^{\infty} \mathcal{H}^d(X_k).$$

We can now take the limit as $\epsilon \to 0$ on the left-hand side to obtain

$$\mathcal{H}^d\left(\bigcup_{k=1}^{\infty} X_k\right) \le \sum_{k=1}^{\infty} \mathcal{H}^d(X_k)$$

as claimed.

13.6 The map L taking $e^{(i)}$ into $v^{(i)}$ $(1 \le i \le n)$ is given by

$$L = \sum_{k=1}^{n} v^{(k)} (e^{(k)})^T,$$

and since $e_i^{(k)} = \delta_{ik}$, the components of L are $L_{ij} = v_i^{(j)}$:

$$(L^T L)_{ij} = v_k^{(i)} v_k^{(j)} = v^{(i)} \cdot v^{(j)} = M_{ij}.$$

13.7 M is real and symmetric since

$$M_{ij} = \delta x^{(i)} \cdot \delta x^{(j)}.$$

It follows that its eigenvalues λ_j are real, and one can find an orthonormal set of eigenvectors $e^{(k)}$ with

$$M e^{(k)} = \lambda_k e^{(k)}.$$

To show that $\lambda_k > 0$, consider

$$\lambda_k = {e^{(k)}}^T M e^{(k)} = e_i^{(k)} \delta x_s^{(i)} \delta x_s^{(j)} e_j^{(k)} = |v^{(k)}|^2 \geq 0,$$

where $v^{(k)}$ is the vector given by its components

$$v_s^{(k)} = e_i^{(k)} \delta x_s^{(i)}.$$

If $v^{(k)} = 0$ then the two different initial conditions

$$\delta x(0) = 0 \qquad \text{and} \qquad \delta x(0) = \sum_{i=1}^{n} e_i^{(k)} \delta x^{(i)}$$

have the same solution at time t, contradicting uniqueness. So all the eigenvalues are strictly positive.

13.8 Writing M as

$$M = \sum_{j=1}^{n} \lambda_j e_j e_j^T,$$

we have

$$\log M = \sum_{j=1}^{n} \log \lambda_j e_j e_j^T.$$

Clearly

$$\mathrm{Tr}[\log M] = \sum_{j=1}^{n} \log \lambda_j, \tag{S13.1}$$

and since

$$\det M = \prod_{j=1}^{n} \lambda_j$$

the required result follows immediately.

Since $\mathrm{Tr}[\log M]$ is given by (S13.1), we have

$$\frac{d}{dt} \mathrm{Tr}[\log M] = \sum_{i=1}^{n} \frac{\dot{\lambda}_i}{\lambda_i}.$$

The right-hand side of (13.34) is

$$\begin{aligned}
\sum_{i=1}^{n} \left(e_i, M^{-1} \frac{dM}{dt} e_i \right) \\
&= \sum_{i=1}^{n} \left(e_i, \left[\sum_{j=1}^{n} \lambda_j^{-1} e_j e_j^T \sum_{k=1}^{n} \left(\dot{\lambda}_k e_k e_k^T + \lambda_k \dot{e}_k e_k^T + \lambda_k e_k \dot{e}_k^T \right) \right] e_i \right) \\
&= \sum_{i=1}^{n} \left(\lambda_i^{-1} e_i^T \left[\sum_{k=1}^{n} \left(\dot{\lambda}_k e_k e_k^T + \lambda_k \dot{e}_k e_k^T + \lambda_k e_k \dot{e}_k^T \right) \right] e_i \right) \\
&= \sum_{i=1}^{n} \left(\lambda_i^{-1} \left[\dot{\lambda}_i e_i^T + \lambda_i \dot{e}_i^T + \sum_{k=1}^{n} \lambda_k (e_i, \dot{e}_k) e_k^T \right] e_i \right) \\
&= \sum_{i=1}^{n} \left[\lambda_i^{-1} \dot{\lambda}_i + (\dot{e}_i, e_i) + (e_i, \dot{e}_i) \right] \\
&= \sum_{i=1}^{n} \lambda_i^{-1} \dot{\lambda}_i,
\end{aligned}$$

since $\frac{d}{dt}(e_i, e_i) = 0$.

13.9 Since the eigenvalues are proportional to the sums of squares of m integers, we will have reached the eigenvalue mk^2 once we have taken k^m combinations of integers. Thus

$$\lambda_{k^m} = Cmk^2,$$

and so if $k^m < n < (k+1)^m$ we obtain

$$Cmk^2 \leq \lambda_n \leq Cm(k+1)^m.$$

We now have

$$k < n^{1/m} < (k+1)$$

and so

$$\tfrac{1}{2} n^{1/m} < k < k+1 < 2n^{1/m}.$$

This gives

$$cn^{2/m} \leq \lambda_n \leq Cn^{2/m},$$

as required.

13.10 Taking the inner product of (13.27) with U we obtain

$$\tfrac{1}{2}\frac{d}{dt}|U|^2 + \nu\|U\|^2 = -b(U, u, U),$$

and so

$$\tfrac{1}{2}\frac{d}{dt}|U|^2 + \nu\|U\|^2 \le k|U|\|U\||u|.$$

Using Young's inequality and rearranging we get

$$\frac{d}{dt}|U|^2 + \nu\|U\|^2 \le C|U|^2. \tag{S13.2}$$

That bounded sets in L^2 are mapped into bounded sets in L^2 follows by neglecting the term in $\|U\|^2$ and applying Gronwall's inequality (Lemma 2.8),

$$|U(t)|^2 \le e^{Ct}|U(0)|^2 = e^{Ct}|\xi|^2. \tag{S13.3}$$

To show that we in fact obtain a bounded set in H^1, we first return to (S13.2) and integrate between $t/2$ and t to obtain

$$\nu\int_{t/2}^{t}\|U(s)\|^2\,ds \le C\int_{t/2}^{t}|U(s)|^2\,ds + |U(t/2)|^2 \le C(t)|U(t/2)|^2, \tag{S13.4}$$

using (S13.3). Now we take the inner product of (13.27) with AU, which gives

$$\tfrac{1}{2}\frac{d}{dt}\|U\|^2 + \nu|AU|^2 = -b(u, U, AU) - b(U, u, AU).$$

Using (9.27) we obtain

$$\tfrac{1}{2}\frac{d}{dt}\|U\|^2 + \nu|AU|^2 \le k\big(|u|^{1/2}\|u\|^{1/2}\|U\|^{1/2}|AU|^{3/2} + |U|^{1/2}\|U\|^{1/2}\|u\|^{1/2}|Au|^{1/2}|AU|\big),$$

and after using Young's inequality and rearranging we have

$$\frac{d}{dt}\|U\|^2 + \nu|AU|^2 \le C\|U\|^2.$$

Expression (S13.4) allows us to use the "uniform Gronwall" trick and find a bound on $\|U\|$ valid for all $t > 0$. Thus $\Lambda(t; u_0)$ is compact for all $t > 0$.

13.11 If we integrate (12.6) between 0 and T we obtain

$$\nu \int_0^T |Au(s)|^2 \, ds \leq \frac{T|f|^2}{\nu} + \|u(0)\|^2.$$

Dividing by T and taking the limit as $T \to \infty$ yields

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T |Au(s)|^2 \, ds \leq \frac{|f|^2}{\nu^2},$$

since there is an absorbing set in V. Therefore

$$\chi \leq \frac{|f|^2}{L^2\nu} = \nu^3 L^{-6} G^2.$$

The only length that can be formed from χ and ν is

$$L_\chi = \left(\frac{\nu^3}{\chi}\right)^{1/6},$$

and this implies (13.35).

Chapter 14

14.1 Since $\mathcal{A}$ is compact, it is bounded and certainly contained in $B(0, r)$ for some $r > 0$. So $N_r(\mathcal{A}) = 1$. We consider

$$S(B(0, r) \cap \mathcal{A}),$$

which by our assumption can be covered by K_0 balls, centred in $\mathcal{A}$, and of radius $r/2$. So

$$N(\mathcal{A}, r/2) = K_0.$$

Now consider each one of the balls in this covering, and apply our assumption again to show that

$$S(B(a_i, r/2) \cap \mathcal{A})$$

can be covered by K_0 balls of radius $r/4$, so that

$$N(\mathcal{A}, r/4) = K_0^2.$$

Iterating this argument, we can see that

$$N(\mathcal{A}, 2^{-k} r) = K_0^k.$$

So therefore, using the result of Exercise 13.2, we have

$$\begin{aligned} d_f(\mathcal{A}) &= \lim_{k\to\infty} \frac{\log(N(\mathcal{A}, 2^{-k}r))}{\log(2^k)} \\ &\leq \frac{k \log K_0}{k \log 2} \\ &\leq n_0 \frac{\log \alpha}{\log 2}, \end{aligned}$$

precisely (14.33).

14.2 We have, for any $u \in D(A^{1/2})$,

$$\|u\|^2 = a(u, u) = (A^{1/2}u, A^{1/2}u) = |A^{1/2}u|.$$

Expanding p in terms of the eigenfunctions of A gives

$$p = \sum_{j=1}^{n} (p, w_j) w_j,$$

and so

$$\|p\|^2 = \sum_{j=1}^{n} \lambda_j |(p, w_j)|^2 \leq \lambda_n |p|^2.$$

Similarly,

$$\|q\|^2 = \sum_{j=1}^{n} \lambda_j |(q, w_j)|^2 \geq \lambda_{n+1} |q|^2.$$

The other two inequalities in the exercise follow easily from these.

14.3 Differentiating Φ gives

$$\frac{d\Phi}{dt} = \exp(\lambda a / C(a+b)) \left[\frac{da}{dt} \left(1 - \frac{\lambda b}{C(a+b)} \right) + \frac{db}{dt} \left(1 + \frac{\lambda a}{C(a+b)} \right) \right].$$

Since we have (14.34), the coefficient of da/dt is negative, whereas the coefficient of db/dt is positive. It follows that we can substitute in the inequalities for da/dt and db/dt, which gives $d\Phi/dt \leq 0$.

14.4 Write

$$|u(x) - u(y)| \leq \sum_{k \in \mathbb{Z}^2} |e^{2\pi i k\cdot x/L} - e^{2\pi i k \cdot y/L}||c_k|,$$

and use (14.35) to deduce that

$$|u(x) - u(y)| \leq C|x-y|^{1/2} \sum_{k\in\mathbb{Z}^2} |c_k||k|^{1/2}$$
$$\leq C|x-y|^{1/2} \left(\sum_{k\in\mathbb{Z}^2} (1+|k|^4)|c_k|^2 \right)^{1/2} \left(\sum_{k\in\mathbb{Z}^2} \frac{|k|}{(1+|k|^4)} \right)^{1/2}$$
$$\leq C\|u\|_{H^2}|x-y|^{1/2}.$$

[$\sum_{k\in\mathbb{Z}^2} |k|/(1+|k|^4)$ is finite.]

14.5 Since (6.14) shows that $\|u\|_{H^2} = C|Au|$ for $u \in D(A)$, we can use the result of the previous exercise to deduce that

$$|u(x) - u(y)| \leq c|Au||x-y|^{1/2}.$$

Expression (14.36) follows immediately from this and the definitions of $d(\mathcal{N})$ and $\eta(u)$.

14.6 Choose $\epsilon > 0$. Then there exists a T such that $b(t) \leq \epsilon/2$ for all $t \geq T$. Hence for $t \geq T$,

$$\frac{dX}{dt} + aX \leq \epsilon/2.$$

By Gronwall's inequality (Lemma 2.8),

$$X(T+t) \leq X(T)e^{-at} + \epsilon/2,$$

and so choosing τ large enough that

$$ke^{-a\tau} < \epsilon/2,$$

we have

$$X(t) \leq \epsilon \qquad \text{for all} \qquad t \geq T+\tau,$$

so that $X(t) \to 0$.

14.7 Using the bound on b given in (9.25), we can write

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \nu|Aw|^2 \leq \|w\|_\infty \|w\||Au|$$
$$\leq [\eta(w) + cd(\mathcal{N})^{1/2}|Aw|]\|w\||Au|$$
$$\leq \eta(w)\|w\||Au| + cd(\mathcal{N})^{1/2}\lambda_1^{-1/2}|Aw|^2|Au|,$$

using (14.36), and therefore

$$\tfrac{1}{2}\frac{d}{dt}\|w\|^2 + \left[\nu - c\lambda_1^{-1/2} d(\mathcal{N})^{1/2}|Au|\right]\lambda_1\|w\|^2 \le \eta(w)\|w\||Au|.$$

Now, we know that $\mathcal{A}$ is bounded in V and $D(A)$, so that

$$\tfrac{1}{2}\frac{d}{dt}\|w\|^2 + [\nu - c\lambda^{1/2}\rho_A d(\mathcal{N})^{1/2}]\lambda_1\|w\|^2 \le 2\rho_V\rho_A\eta(w).$$

Now, choose δ such that

$$\mu = \nu - c\lambda_1^{1/2}\rho_A\delta^{1/2} > 0.$$

Then we have, for $d(\mathcal{N}) < \delta$,

$$\tfrac{1}{2}\frac{d}{dt}\|w\|^2 + \mu\|w\|^2 \le 2\rho_V\rho_A\eta(w). \qquad \text{(S14.1)}$$

By assumption, we know that $\eta(w) \to 0$, and since the attractor is bounded in V we have $\|w(t)\|^2 \le 4\rho_V^2$. The result of the previous exercise applied to (S14.1) now shows (14.37).

14.8 (i) Take the inner product of (14.38) with $q_n = Q_n u$ to obtain

$$\tfrac{1}{2}\frac{d}{dt}|q_n|^2 + (Au, q_n) = (F(u), q_n).$$

Now, notice that

$$|(Au, q_n)| = |(Aq_n, q_n)| \ge \lambda_{n+1}|q_n|^2,$$

and so

$$\tfrac{1}{2}\frac{d}{dt}|q_n|^2 + \lambda_{n+1}|q_n|^2 \le C_0|q_n|,$$

from which, using the result of Exercise 2.5, we see that

$$\frac{d}{dt_+}|q_n| \le -\lambda_{n+1}|q_n| + C_0,$$

which gives

$$|Q_n u(t)| \le \frac{C_0}{\lambda_{n+1}} + |Q_n u(0)|, \qquad \text{(S14.2)}$$

using the Gronwall lemma (Lemma 2.8).

(ii) Writing $p(t) = P_n u(t)$ and $q(t) = Q_n u(t)$, p solves the equation

$$dp/dt + Ap = P_n F(p+q).$$

Thus the equation for $w = p - p_n$ is

$$dw/dt + Aw = P_n F(p_n) - P_n F(p+q).$$

Taking the inner product with w and using the Lipschitz property of F gives

$$\tfrac{1}{2}\frac{d}{dt}|w|^2 + \|w\|^2 \le C_1|w|^2 + C_1|q||w|.$$

Hence

$$\frac{d}{dt_+}|w| \le C_1|w| + C_1|q|,$$

and so, using the bound in (S14.2) and the Gronwall lemma as above we obtain

$$|P_n u(t) - p_n(t)| \le C_1^{-1}\left[\frac{C_0}{\lambda_{n+1}} + |Q_n u(0)|\right]e^{C_1 t}.$$

Combining this with (S14.2) yields

$$|u(t) - p_n(t)| \le C_1^{-1}\left[\frac{C_0}{\lambda_{n+1}} + |Q_n u(0)|\right](C_1 + e^{C_1 t}),$$

and since we know that $\lambda_{n+1} \to \infty$ and $|Q_n u(0)| \to 0$ as $n \to \infty$, it follows that $p_n(t)$ converges to $u(t)$ as claimed.

Chapter 15

15.1 For any point $v \in H$,

$$\operatorname{dist}(v, \mathcal{M})^2 = \inf_{p\in PH}\left(|Pv - p|^2 + |Qv - \phi(p)|^2\right)$$

and

$$\begin{aligned}|Qv - \phi(Pv)|^2 &= |Qv - \phi(p) + \phi(p) - \phi(Pv)|^2 \\ &\le 2|Qv - \phi(p)|^2 + 2|\phi(p) - \phi(Pv)|^2 \\ &\le 2|Qv - \phi(p)|^2 + 2l^2|Pv - p|^2 \\ &\le c^2\left(|Qv - \phi(p)|^2 + |Pv - p|^2\right)\end{aligned}$$

for all $p \in PH$, where $c^2 = 2\max(l^2, 1)$. Therefore

$$|Qv - \phi(Pv)| \le c\,\mathrm{dist}(v, \mathcal{M}).$$

The other implication is obvious.

15.2 Using Proposition 15.3 we see that the attractor lies in the graph of some Lipschitz function $\Phi : P_nH \to Q_nH$. We can therefore project the dynamics on $\mathcal{A}$ onto P_nH by writing

$$dp/dt + Ap = P_nF(p + \Phi(p)). \tag{S15.1}$$

It is easy to show that (S15.1) is a Lipschitz ODE on P_nH, since

$$\begin{aligned}|P_nF(p+\Phi(p)) - P_nF(\overline{p}+\Phi(\overline{p}))| &\le |F(p+\Phi(p)) - F(\overline{p}+\Phi(\overline{p}))| \\ &\le C|p + \Phi(p) - \overline{p} - \Phi(\overline{p})| \\ &\le C\big(|p - \overline{p}| + |\Phi(p) - \Phi(\overline{p})|\big) \\ &\le 2C|p - \overline{p}|.\end{aligned}$$

We know that if $u(t)$ is a solution in $\mathcal{A}$ then $p(t) = P_nu(t)$ is a solution of (S15.1) lying in $P_n\mathcal{A}$. Since (S15.1) is Lipschitz its solutions are unique, and so in particular $P_n\mathcal{A}$ is an invariant set. Thus (S15.1) is a finite-dimensional system that reproduces the dynamics on $\mathcal{A}$. [The advantage of the inertial form over (S15.1) is that $P_n\mathcal{A}$ is the attractor of the finite-dimensional system, not just an invariant set.]

15.3 Since $F = 0$ outside $B(0, \rho)$,

$$\Sigma_{t_0} \subset S(t_0)\{u : u \in P_nH : \rho \le |u| \le \rho e^{\lambda_{n+1}t_0}\}.$$

The cone invariance part of the strong squeezing property then shows that for any two points u_1 and u_2 in Σ_{t_0} we must have

$$|Q_n(u_1 - u_2)| \le |P_n(u_1 - u_2)|.$$

If we write

$$\Sigma = \bigcup_{0 \le t < \infty} S(t)\Gamma$$

then the function Φ defined by

$$\Phi(P_nu) = Q_nu \qquad \text{for all} \qquad u \in \overline{\Sigma}$$

is Lipschitz on its domain of definition, $P_n\overline{\Sigma} = P_n B(0, \rho)$. Clearly $\overline{\Sigma}$ is positively invariant, and so $\mathcal{M}$ is invariant.

To show that $\mathcal{A} \subset \mathcal{M}$, suppose that $u \in \mathcal{A}$ and $v \in \mathcal{M}$ with $P_n u = P_n v$ but $Q_n u \neq Q_n v$. Then, using the invariance of $\overline{\Sigma}$ and $\mathcal{A}$, we have $u = S(t)u_t$ with $u_t \in \mathcal{M}$, and $v = S(t)v_t$ with $v_t \in \mathcal{A}$. Thus

$$\begin{aligned} |Q_n(u - v)| &\leq |Q_n(u_t - v_t)|e^{-kt} \\ &\leq 2\rho e^{-kt}, \end{aligned} \tag{S15.2}$$

since both $\mathcal{A}$ and $\overline{\Sigma}$ are subsets of $B(0, \rho)$. Since (S15.2) holds for all $t \geq 0$, we must have $Q_n u = Q_n v$. Thus $u = v$ and $\mathcal{A} \subset \mathcal{M}$ as claimed.

15.4 We have

$$\begin{aligned} 135 &= 1^2 + 2^2 + 3^2 + 11^2, \\ 136 &= 6^2 + 10^2, \\ 137 &= 4^2 + 11^2, \\ 138 &= 1^2 + 3^2 + 8^2 + 8^2, \end{aligned}$$

all as sums of (at most) four squares.

15.5 (i) If $u(t)$ is a solution of (15.24), then $p(t) = P_n u(t)$ is the solution of the equation

$$dp/dt + Ap = P_n F(p(t) + q(t)).$$

Since F is Lipschitz, it follows that

$$\begin{aligned} |P_n F(p(t)+q(t)) - P_n F(p(t)+\Phi(p(t)))| &\leq C_1|q(t) - \Phi(p(t))| \\ &\leq C_1 C e^{-kt}, \end{aligned}$$

where the result of Exercise 15.1 has been used.

(ii) Let $\overline{u}(t) = \overline{p}(t) + \Phi(\overline{p}(t))$. Then $\overline{u}(t) \in \mathcal{M}$, so we just have to show the exponential convergence in (15.26). To do this, we write

$$\begin{aligned} |u(t) - \overline{u}(t)| &\leq |p(t) + q(t) - p(t) - \Phi(p(t))| \\ &\quad + |p(t) + \Phi(p(t)) - \overline{p}(t) - \Phi(\overline{p}(t))| \\ &\leq |q(t) - \Phi(p(t))| + 2|p(t) - \overline{p}(t)| \\ &\leq Ce^{-kt} + 2De^{-kt} = Me^{-Kt}, \end{aligned}$$

where we have used the result of Exercise 15.1 again and the Lipschitz property of Φ.

Chapter 16

16.1 $\omega(r)$ is clearly well defined, since the set

$$\{(x, y) \in X \times X : |x - y| \leq r\}$$

is a compact subset of $X \times X$. The convexity property follows easily, since

$$\begin{aligned}\omega(r+s) &= \sup_{|x-z|\leq r+s} |f(x) - f(z)| \\ &\leq \sup_{|x-y|\leq r,\ |y-z|\leq s} |f(x) - f(y)| + |f(y) - f(z)| \\ &\leq \sup_{|x-y|\leq r} |f(x) - f(y)| + \sup_{|y-z|\leq s} |f(x) - f(y)| \\ &= \omega(r) + \omega(s),\end{aligned}$$

where to prevent too clumsy notation we have assumed throughout that $x, y, z \in X$.

16.2 (i) X can be covered by $N(X, \epsilon)$ balls of radius ϵ and, in particular, lies within ϵ of the space spanned by the centres of these balls. Therefore $d(X, \epsilon) \leq N(X, \epsilon)$, and the inequality follows.

(ii) Simply choose any open subset $\mathcal{O}$ in $\mathbb{R}^n$. Then $d_f(\mathcal{O}) = n$ but since $\mathcal{O} \subset \mathbb{R}^n$ we must have $\tau(\mathcal{O}) = 0$.

16.3 Consider the projection P_n onto the space spanned by the first n eigenfunctions of A,

$$P_n u = \sum_{j=1}^{n} (u, w_j) w_j,$$

and its orthogonal complement $Q_n = I - P_n$. Then

$$\begin{aligned}|u - P_n u| &= |Q_n u| \\ &= |Q_n A^{-s/2} A^{s/2} u| \\ &\leq \|Q_n A^{-s/2}\|_{\text{op}} |A^{s/2} u| \\ &\leq \lambda_{n+1}^{-s/2} \|u\|_{H^s} \\ &\leq C n^{-2s/m}\end{aligned}$$

for some constant C. Clearly,

$$\log d(X, \epsilon) \leq \frac{\log \epsilon}{-2s/m} + \frac{\log C}{2s/m},$$

and so one obtain (16.23). If X is bounded in $D(A^r)$ for any r then it follows from (16.23) that $\tau(X) = 0$, and so one can obtain any θ in the range

$$0 < \theta < 1 - \frac{2d_f(X)}{k}.$$

We can now obtain any $\theta < 1$ by choosing k large enough.

16.4 Write $w = u - v$ for $u, v \in \mathcal{A}$. If A is Lipschitz continuous from $\mathcal{A}$ into H then

$$|Aw| \le L|w|$$

for some L. Now split $w = P_n w + Q_n w$, and observe that we have both

$$|Aw|^2 = |A(P_n w + Q_n w)|^2 = |A(P_n w)|^2 + |A(Q_n w)|^2 \ge \lambda_{n+1}^2 |Q_n w|^2$$

and

$$|Aw|^2 \le L^2|w|^2 \le L^2|P_n w|^2 + L^2|Q_n w|^2.$$

Since $\lambda_n \to \infty$ as $n \to \infty$, we can choose n large enough that $\lambda_{n+1} > L$, and then write

$$\lambda_{n+1}^2 - L^2|Q_n w|^2 \le L^2|P_n w|^2,$$

that is,

$$|Q_n w| \le \left(\frac{L^2}{\lambda_{n+1}^2 - L^2}\right)^{1/2} |P_n w|.$$

It follows that we can define $\Phi(P_n u) = Q_n u$ uniquely for each $u \in \mathcal{A}$, and then

$$|\Phi(p_1) - \Phi(p_2)| \le \left(\frac{L^2}{\lambda_{n+1}^2 - L^2}\right)^{1/2} |p_1 - p_2|,$$

so that (cf. Proposition 15.3) the attractor is a subset of a Lipschitz graph over $P_n H$.

16.5 Since X is the attractor for $\dot{x} = g(x)$, given $\epsilon > 0$, there exists a $\delta > 0$ such that if $x(0) \in N(X, \delta)$ then the solution $x(t)$ of $\dot{x} = g(x)$ remains within $N(X, \epsilon)$ for all $t \ge 0$.

Define $\tilde{f}(x)$ on a closed subset of $\mathbb{R}^n$ by

$$\tilde{f}(x) = \begin{cases} f(x), & \operatorname{dist}(x, X) \leq \delta/4, \\ 0, & \operatorname{dist}(x, X) \geq \delta/2. \end{cases}$$

Since $\tilde{f}$ is Lipschitz on its domain of definition, it can be extended using Theorem 16.4 to a function $F(x)$ that is Lipschitz on $\mathbb{R}^n$. Now consider

$$\dot{x} = F(x) + g(x). \tag{S16.1}$$

Clearly X is an invariant subset for (S16.1), since $F(x) + g(x) = F(x)$ on X. To show that the attractor of (S16.1) lies within an $N(X, \epsilon)$ it suffices to show that $N(X, \epsilon)$ is absorbing. This follows from the choice of δ and the fact that $F(x) + g(x) = g(x)$ outside $N(X, \delta/2)$.

Chapter 17

17.1 Integrating (17.3) between 0 and L and using the periodic boundary conditions gives

$$\int_0^L |Du|^2 = -\int_0^L u D^2 u \, dx,$$

which implies (17.4) after an application of the Cauchy–Schwarz inequality. For $u \in \dot{H}^2_{\mathrm{p}}$ the result follows by finding a sequence $\{u_n\} \in \dot{C}^2_{\mathrm{p}}$ that converges to u in the norm of H^2_{p}.

17.2 Multiplying (17.5) by a function ϕ in $\dot{C}^2_{\mathrm{p}}$ and integrating by parts twice gives

$$\int_0^L (D^2 u)(D^2 \phi) \, dx = \int_0^L f(x)\phi(x) \, dx. \tag{S17.1}$$

Define a bilinear form $a(u, v) : \dot{H}^2_{\mathrm{p}} \times \dot{H}^2_{\mathrm{p}} \to \mathbb{R}$ by

$$a(u, v) = \int_0^L (D^2 u)(D^2 v) \, dx,$$

and then, using the density of $\dot{C}^2_{\mathrm{p}}$ in $\dot{H}^2_{\mathrm{p}}$, we see that (S17.1) becomes (17.6).

17.3 Since $a(u, v)$ is equivalent to the inner product on $\dot{H}^2_{\rm p}$ (by the general Poincaré inequality from Exercise 5.4), we can use the Riesz representation theorem to deduce the existence of a unique solution $u \in \dot{H}^2_{\rm p}$ of (17.6) for any $f \in H^{-2}$.

In particular if $f \in \dot{L}^2$ then $u \in \dot{H}^2_{\rm p}$, which is a compact subset of $\dot{L}^2$, using the Rellich–Kondrachov compactness theorem (Theorem 5.32). It follows that the inverse of A is compact, and A itself is clearly symmetric. We can therefore apply Corollary 3.26 to deduce that A has an orthonormal set of eigenfunctions $\{w_j\}$ that form a basis for $\dot{L}^2$.

17.4 The orthogonality property (17.8) follows easily, since for $u \in \dot{C}^2_{\rm p}$,

$$b(u, u, u) = \int_0^L u(x)^2 \frac{du}{dx}\, dx = \tfrac{1}{3} \int_0^L \frac{d}{dx} u(x)^3\, dx = 0,$$

using the periodic boundary conditions. The result follows for all $u \in \dot{H}^2_{\rm p}$ by taking limits. Similarly for the cyclic equality, after an integration by parts, we have

$$\int_0^L u v_x w\, dx = -\int_0^L (uw)_x v\, dx = -\int_0^L v w_x u + w u_x v\, dx.$$

The inequalities in (17.10) follow from the estimate

$$\int uvw\, dx \le \|u\|_\infty |v||w| \le |Du||v||w|,$$

since $H^1 \subset C^0$ on a one-dimensional domain (Theorem 5.31).

17.5 Taking the inner product of (17.12) with u_n gives

$$\tfrac{1}{2}\frac{d}{dt}|u_n|^2 + a(u_n, u_n) + (D^2 u_n, u_n) + (P_n B(u_n, u_n), u_n) = 0.$$

Since

$$(P_n B(u_n, u_n), u_n) = (B(u_n, u_n), P_n u_n) = (B(u_n, u_n), u_n) = 0$$

by (17.8), we obtain

$$\tfrac{1}{2}\frac{d}{dt}|u_n|^2 + |D^2 u_n|^2 = |Du_n|^2.$$

Using (17.4) we have

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}|u_n|^2 + |D^2 u_n|^2 &\le |u_n||D^2 u_n| \\ &\le \tfrac{1}{2}|u_n|^2 + \tfrac{1}{2}|D^2 u_n|^2,\end{aligned}$$

and so

$$\frac{d}{dt}|u_n|^2 + |D^2 u_n|^2 \leq |u_n|^2. \tag{S17.2}$$

Dropping the term in $|D^2 u_n|^2$ and integrating we get

$$|u_n(t)|^2 \leq e^t |u_n(0)|^2,$$

so clearly

$$u_n \qquad \text{is uniformly bounded in} \qquad L^\infty(0, T; \dot{L}^2).$$

Integrating (S17.2) as it stands then gives

$$|u_n(t)|^2 + \int_0^t |D^2 u_n(s)|^2 \, ds \leq \int_0^t |u_n(s)|^2 \, ds + |u_n(0)|$$

and in particular shows that

$$u_n \qquad \text{is uniformly bounded in} \qquad L^2\big(0, T; \dot{H}^2_{\mathrm{p}}\big).$$

It follows from these estimates, the equality

$$du_n/dt = -Au - D^2 u - B(u, u),$$

and Poincaré's inequality (17.2) that

$$du_n/dt \qquad \text{is uniformly bounded in} \qquad L^2(0, T; H^{-2}),$$

and we have obtained the bounds in (17.13).

17.6 Extracting subsequences from the $\{u_n\}$ and relabelling as necessary we find a u such that

$$u \in L^2\big(0, T; \dot{H}^2_{\mathrm{p}}\big) \cap L^\infty(0, T; \dot{L}^2) \quad \text{with} \quad du/dt \in L^2(0, T; H^{-2}),$$

and

$$\begin{aligned} u_n &\rightharpoonup u && \text{in} && L^2\big(0, T; \dot{H}^2_{\mathrm{p}}\big), \\ u_n &\overset{*}{\rightharpoonup} u && \text{in} && L^\infty(0, T; \dot{L}^2), \\ du_n/dt &\overset{*}{\rightharpoonup} du/dt && \text{in} && L^2(0, T; H^{-2}). \end{aligned}$$

We can also use the compactness theorem (Theorem 8.1) to find a subsequence with the additional strong convergence

$$u_n \to u \qquad \text{in} \qquad L^2\big(0, T; \dot{H}^1_{\mathrm{p}}\big),$$

since $\dot{H}^2_{\rm p} \subset\subset \dot{H}^1_{\rm p} \subset H^{-2}$. It is simple to show the weak-* convergence in $L^2(0, T; H^{-2})$ of all the terms in the equation, except for the non-linear term. For this we need the strong convergence in $L^2(0, T; \dot{H}^1_{\rm p})$ and the uniform bound on u_n in $L^\infty(0, T; \dot{L}^2)$. We need to show that

$$\int_0^T b(u_n, u_n, v)\, dt \to \int_0^T b(u, u, v)\, dt \quad \text{for all} \quad v \in L^2\big(0, T; \dot{H}^2_{\rm p}\big).$$

Using (17.9) we write

$$\begin{aligned} b(u_n, u_n, v) - b(u, u, v) &= b(u_n - u, u_n, v) + b(u, u_n - u, v) \\ &= -b(u_n, v, u_n - u) - b(v, u_n - u, u_n) + b(u, u_n - u, v), \end{aligned}$$

and then for the first term

$$\begin{aligned} \int_0^T |b(u_n, v, u_n - u)|\, dt &\le k \int_0^T |u_n||D^2 v||u_n - u|\, dt \\ &\le k\|u_n\|_{L^\infty(0,T;\dot{L}^2)}\|v\|_{L^2(0,T;\dot{H}^2_{\rm p})}\|u_n - u\|_{L^2(0,T;\dot{L}^2)} \\ &\to 0, \end{aligned}$$

and for the second and third terms

$$\begin{aligned} \int_0^T |b(v, u_n - u, u_n)|\, dt &\le k \int_0^T |Dv||D(u_n - u)||u_n|\, dt \\ &\le k\|u_n\|_{L^\infty(0,T;\dot{L}^2)}\|v\|_{L^2(0,T;\dot{H}^2_{\rm p})}\|u_n - u\|_{L^2(0,T;\dot{H}^1_{\rm p})} \\ &\to 0, \end{aligned}$$

giving the required convergence. That

$$P_n B(u_n, u_n) \overset{*}{\rightharpoonup} B(u, u)$$

follows as in Exercise 9.5.

Finally, the continuity of u into $\dot{L}^2$ follows from the generalisation of Theorem 7.2 discussed after its formal statement in Chapter 7.

17.7 The equation for the difference w of two solutions, $w = u - v$, is

$$w_t + w_{xxxx} + w_{xx} + wu_x + vw_x = 0.$$

Taking the inner product with w we obtain

$$\tfrac{1}{2}\frac{d}{dt}|w|^2 + |D^2 w|^2 - |Dw|^2 = -b(w, u, w) - b(v, w, w).$$

Estimating the terms on the right-hand side by using (17.10) we have

$$\tfrac{1}{2}\frac{d}{dt}|w|^2 + |D^2 w|^2 \le |Dw|^2 + |D^2 u||w|^2 + |v||Dw|^2.$$

Using (17.4) and Young's inequality gives

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}|w|^2 + |D^2 w|^2 &\le (1 + |v|)|w||D^2 w| + |D^2 u||w^2| \\ &\le \tfrac{1}{2}|D^2 w|^2 + C(1 + |D^2 u| + |v|^2)|w|^2,\end{aligned}$$

and so

$$\frac{d}{dt}|w|^2 + |D^2 w|^2 \le C(1 + |D^2 u| + |v|^2)|w|^2. \tag{S17.3}$$

Neglecting the term in $|D^2 w|^2$ and integrating from 0 to t shows (17.15). Since $u, v \in L^2(0, T; \dot{H}^2_{\rm p})$, it follows that $w(t) = 0$ for all t if $w(0) = 0$, which gives uniqueness.

17.8 Choosing $\alpha = 6$ we have

$$\frac{d}{dt}|v|^2 + \tfrac{1}{2}|D^2 v|^2 + 2|v|^2 \le \tfrac{1}{2}|g|^2, \tag{S17.4}$$

and so in particular

$$\frac{d}{dt}|v|^2 \le -2|v|^2 + \tfrac{1}{2}|g|^2.$$

The Gronwall inequality (Lemma 2.8) now shows that

$$|v(t)|^2 \le |v(0)|^2 e^{-2t} + \tfrac{1}{4}|g|^2(1 - e^{-2t}). \tag{S17.5}$$

Since $u = \phi + v$ and $\phi \in \dot{C}^\infty_{\rm p}$ is constant, it follows that there is an absorbing set for $u(t)$ in L^2.

We can also obtain from (S17.4) a bound on the integral of $|D^2 v|^2$,

$$\tfrac{1}{2}\int_t^{t+1} |D^2 v(s)|^2\, ds \le \tfrac{1}{2}|g|^2 + |v(t)|^2,$$

or for $|D^2 u|^2$ the bound

$$\int_t^{t+1} |D^2 u(s)|^2\, ds \le |g|^2 + |D^2 \phi|^2 + |v(t)|^2.$$

It follows from (S17.5) that if t is large enough then

$$\int_t^{t+1} |D^2 u(s)|^2\, ds \le M, \tag{S17.6}$$

and we have both bounds in (17.18).

17.9 Taking the inner product of (17.11) with $-D^2u$ we obtain

$$\frac{1}{2}\frac{d}{dt}|Du|^2 + |D^3u|^2 = |D^2u|^2 + b(u, u, D^2u).$$

We now we estimate the right-hand side by using (17.10),

$$\frac{1}{2}\frac{d}{dt}|Du|^2 + |D^3u|^2 \le |D^2u|^2 + |D^2u||Du|^2.$$

Neglecting the term in $|D^3u|^2$ we have

$$\frac{d}{dt}|Du|^2 \le |D^2u|^2 + |D^2u||Du|^2.$$

Note that this is in the form in which the uniform Gronwall lemma of Exercise 11.2 is applicable, since we have a uniform estimate on the integral of $|D^2u|$ provided in (S17.6) above. It follows that there is an absorbing set in $\dot{H}^1_{\mathrm{p}}$.

We have therefore obtained a compact absorbing set in L^2 and proved the existence of a global attractor.

17.10 As in the proof of Theorem 13.20, we consider the equation for $\theta = u - v - U$,

$$\theta_t + \theta_{xxxx} + \theta_{xx} + \theta u_x + w w_x = 0,$$

where $w = u - v$. Taking the inner product with θ yields

$$\frac{1}{2}\frac{d}{dt}|\theta|^2 + |D^2\theta|^2 = |D\theta|^2 - b(\theta, u, \theta) - b(w, w, \theta).$$

Using (17.4) and (17.10) on the right-hand side we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\theta|^2 + |D^2\theta|^2 &\le |\theta||D^2\theta| + |\theta|^2|D^2u| + |Dw|^2|\theta| \\ &\le \tfrac{1}{2}|\theta|^2 + \tfrac{1}{2}|D^2\theta|^2 + |D^2u||\theta|^2 + \tfrac{1}{2}|Dw|^4 + \tfrac{1}{2}|\theta|^2,\end{aligned}$$

and so

$$\frac{d}{dt}|\theta|^2 + |D^2\theta|^2 \le 2(1 + |D^2u|)|\theta|^2 + |Dw|^4.$$

It follows from Gronwall's inequality (Lemma 2.8), since $\theta(0) = 0$, that

$$|\theta(t)|^2 \le k(t)\int_0^t |Dw(s)|^4\, ds,$$

and so, using (17.4), we get

$$|\theta(t)|^2 \leq k \int_0^t |w(s)|^2 |D^2 w(s)|^2 \, ds.$$

Returning to (S17.3),

$$\frac{d}{dt}|w|^2 + |D^2 w|^2 \leq C(1 + |D^2 u| + |v|^2)|w|^2,$$

multiplying both sides by $|w|^2$, and integrating we obtain

$$\int_0^t |w(s)|^2 |D^2 w(s)|^2 \, ds \leq C \int_0^t |w(s)|^4 \, ds + \tfrac{1}{4}|w(0)|^4.$$

Using (17.15) we have

$$\int_0^t |w(s)|^2 |D^2 w(s)|^2 \, ds \leq C(t)|w(0)|^4,$$

and hence

$$|\theta(t)|^2 \leq K(t)|w(0)|^4.$$

The uniform differentiability property now follows.

17.11 To show that $\Lambda(t; u_0)$ is compact take the inner product of (17.19) with U to obtain

$$\tfrac{1}{2}\frac{d}{dt}|U|^2 + |D^2 U|^2 - |DU|^2 + b(U, u, U) + b(u, U, U) = 0.$$

Using the cyclic property (17.9) and the bound in (17.10) we have

$$\tfrac{1}{2}\frac{d}{dt}|U|^2 + |D^2 U|^2 \leq C|DU|^2.$$

Using (17.4) and Young's inequality we end up with

$$\frac{d}{dt}|U|^2 + |D^2 U|^2 \leq C|U|^2. \tag{S17.7}$$

Dropping the term in $|D^2 U|^2$ shows that

$$|U(t)|^2 \leq e^{Ct}|\xi|^2, \tag{S17.8}$$

and integrating between $t/2$ and t shows that (cf. Exercise 13.10)

$$\int_{t/2}^t |D^2 U(s)|^2 \, ds \leq C(t)|U(t/2)|^2. \tag{S17.9}$$

Now take the inner product of (17.19) with $-D^2U$ and obtain

$$\tfrac{1}{2}\frac{d}{dt}|DU|^2 + |D^3U|^2 = |D^2U|^2 + b(U, u, D^2U) + b(u, U, D^2U)$$
$$\leq |D^2U|^2 + |DU||Du||D^2U| + |u||D^2U|^2,$$

by using (17.10). We can use the Poincaré inequality (17.2) and drop the term in $|D^3U|^2$ to give

$$\frac{d}{dt}|DU|^2 \leq C|D^2U|^2.$$

Using (S17.9) and the uniform Gronwall "trick" shows that a bounded set in L^2 becomes a bounded set in H^1, and so $\Lambda(t; u_0)$ is compact for all $t > 0$ as claimed.

17.12 We use (17.4) to estimate the second term on the right-hand side by

$$\sum_{j=1}^{n} |D\phi_j|^2 \leq \sum_{j=1}^{n} |\phi_j||D^2\phi_j| \leq \left(\sum_{j=1}^{n} |\phi_j|^2\right)^{1/2} \left(\sum_{j=1}^{n} |D^2\phi_j|^2\right)^{1/2}.$$

Since the $\{\phi_j\}$ are orthonormal, $|\phi_j|^2 = 1$, giving

$$\sum_{j=1}^{n} |D\phi_j|^2 \leq n^{1/2}\left(\sum_{j=1}^{n} |D^2\phi_j|^2\right)^{1/2} \leq n + \tfrac{1}{4}\sum_{j=1}^{n} |D^2\phi_j|^2.$$

To estimate the final term, we use the Cauchy–Schwarz inequality,

$$\int_0^L \phi_j^2 Du\, dx \leq |\phi_j^2||Du| = \|\phi_j\|_{L^4}^2|Du|$$
$$\leq C|D\phi_j|^2,$$

since $|Du|$ is bounded on $\mathcal{A}$ and $H^1 \subset L^4$. Now, using (17.4), we have

$$\int_0^L \phi_j^2 Du\, dx \leq C|\phi_j||D^2\phi_j|$$
$$\leq C|\phi_j|^2 + \tfrac{1}{4}|D^2\phi_j|^2.$$

Combining these estimates we have

$$\sum_{j=1}^{n}(L\phi_j, \phi_j) \leq -\tfrac{1}{2}\sum_{j=1}^{n} |D^2\phi_j|^2 + Mn.$$

Since the eigenvalues λ_j of $A = D^4$ are proportional to j^4, it follows (cf. final part of the argument in the proof of Lemma 13.17) that

$$\sum_{j=1}^{n} |D^2\phi_j|^2 \geq Cn^5.$$

Therefore we need

$$-Cn^5 + Mn < 0,$$

which occurs provided that $n > (M/C)^{1/4}$. The KSE therefore has a finite-dimensional attractor.

17.13 For $v \in D(A^{1/2})$ we have

$$\begin{aligned}(N(u), v) &= \int_0^L u(Du)v + (D^2u)v\,dx \\ &= -\int_0^L \tfrac{1}{2}u^2 Dv - uD^2v\,dx,\end{aligned}$$

and so

$$|(N(u), v)| \leq \tfrac{1}{2}|u|^2\|Dv\|_{L^\infty} + |u||D^2v|.$$

Since $H^1 \subset L^\infty$ and $D(A^{1/2}) \subset H^2$ then

$$|(N(u), v)| \leq C(|u| + 1)|u||A^{1/2}v|,$$

as required.

17.14 For $w \in D(A^{1/2})$,

$$\begin{aligned}(N(u) - N(v), w) &= \int_0^L (uDu - vDv)w + D^2(u - v)w\,dx \\ &= \int_0^L \tfrac{1}{2}(u^2 - v^2)Dw + (u - v)(D^2w)\,dx,\end{aligned}$$

and so

$$\begin{aligned}|(N(u) - N(v), w)| &\leq (\tfrac{1}{2}(|u + v| + 1)|u|\|Dw\|_{L^\infty} \\ &\leq c(|u + v| + 1)|u||A^{1/2}w|,\end{aligned}$$

where the same embedding results as those given above were used.