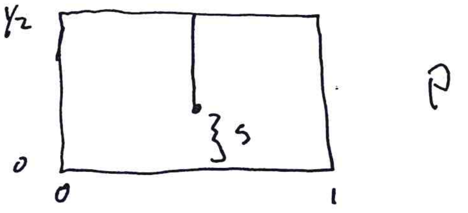
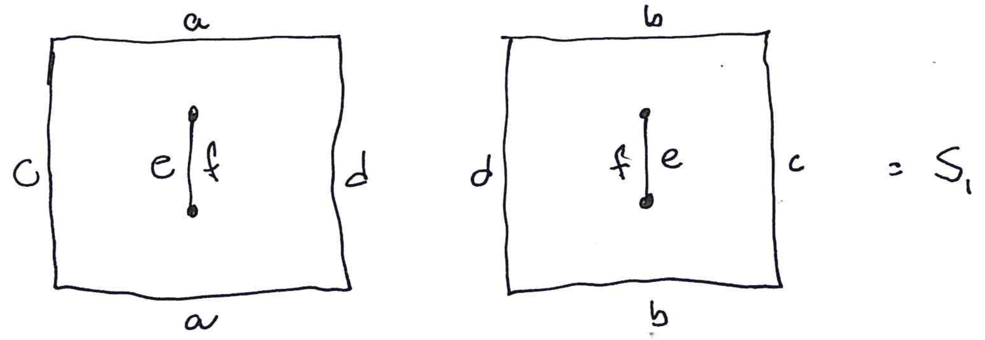
Warwick's webpage has notes on topology of branched covers. ①

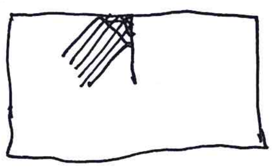
Recall that last time we were in the process of constructing nonergodic directions in the slit torus with s irrational.



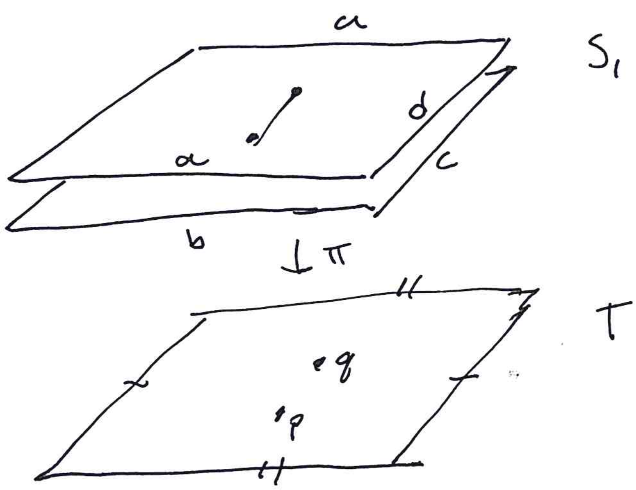
We built the Z-K surface



I should have observed that the cone angle $\pi/2$ vertices unfold to non-singular points on S_1. This means that the billiard flow can be extended continuously to a flow through these vertices. S_1 can be viewed as a double cover of the torus with 2 branch points. At each branch point the cone angle is 4π.



According to our stratum notation $S_1 \in \mathcal{H}(1,1)$. S_1 has genus 2.



$p = (1/2, s)$

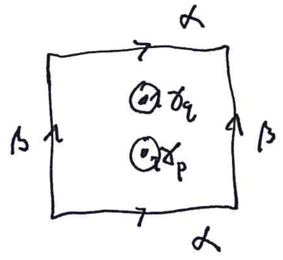
$q = (1/2, 1-s)$

Let ι be the sheet interchange map. On the level of P, ι corresponds to the reflection through the center slit.

We mentioned last time that not all branched covers of the torus are isomorphic and that they are determined by a homomorphism $\chi: \pi_1(T - \{p,q\}) \to \mathbb{Z}_2$ $(= \mathbb{Z}/2\mathbb{Z})$.

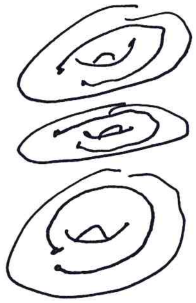
Let's calculate this homomorphism for S_1.

χ_{S_1} is determined by its values on α, β, γ_p and γ_q.



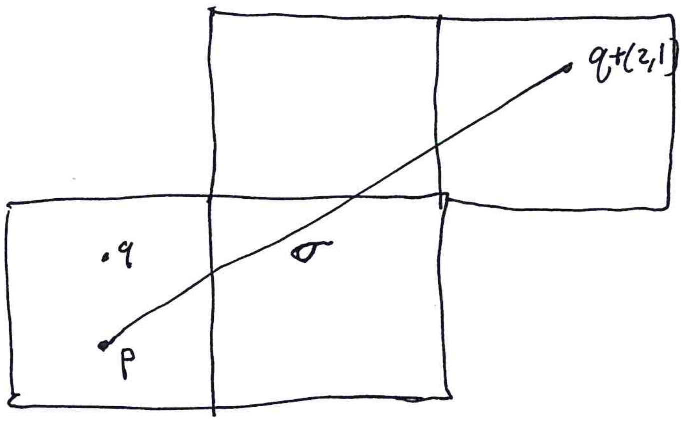
We have
$$\chi(\alpha) = 1$$
$$\chi(\beta) = 0$$
$$\chi(\gamma_q) = 1$$
$$\chi(\gamma_p) = 1$$

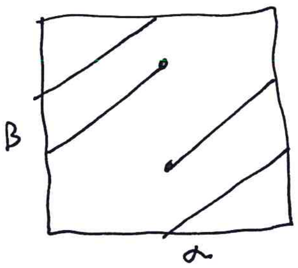
Let $\sigma(m,n)$ be a slit on the torus which starts at $p=(\frac{1}{2},s)$ and ends at a translate of $q=(\frac{1}{2},1-s)+(m,n)$. When is the flow in the direction of σ non-minimal? This will be the case when, if we lift σ up to two saddle connections σ_0 and σ_1 in S_1, the loop formed by $\sigma_0 \cup \sigma_1$ disconnects S_1. Now it is easy to build some branched cover S'_σ in which this loop does disconnect: we



simply take two copies of T both slit along σ and glue them together. The question of whether σ disconnects is the question of whether S_1 is isomorphic to S'_σ. To check this we compare χ_{S_1} with $\chi_{S'_\sigma}$.

Example. $\sigma = \sigma(2,1)$



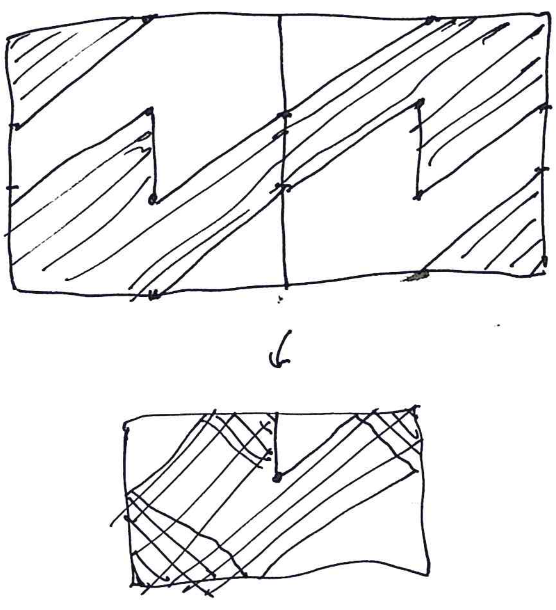


To evaluate χ_{S_σ} on a loop ρ we count the parity of $\# \rho \cap \sigma$ since each time we cross σ we change the labelling on the fiber.

$\chi_{S_\sigma}(\alpha) = 1 \quad \chi_{S_\sigma}(\beta) = 0 \quad \chi_{S_\sigma}(\gamma_p) = 1 \quad \chi_{S_\sigma}(\gamma_q) = 1.$

So S_σ is isomorphic to S_1 and $\sigma(2,1)$ does divide S_1 into two submanifolds A_σ and B_σ. These will be interchanged by ι.

Check



In general for $\sigma = \sigma(m,n)$

$$\chi_{S_\sigma}(\alpha) = n \mod 2 \qquad \chi_{S_\sigma}(\beta) = m \mod 2$$

$$\chi(\gamma_p) = 1 \qquad \chi(\gamma_q) = 1$$

So the flow in the direction of σ is non-minimal when m is even and n is odd. This condition is actually necessary and sufficient.

Now say we have an infinite sequence of saddle connections $\sigma_i = \sigma(m_i, n_i)$ which satisfy our parity condition and so give us partitions A_i, B_i of S_1 invariant in the direction of σ_i. We described last time 3 conditions on these partitions which lead to convergence of directions and convergence of partitions. The key property was $\sum_{i=1}^{\infty} \mu(A_i \triangle A_{i+1}) < \infty$.

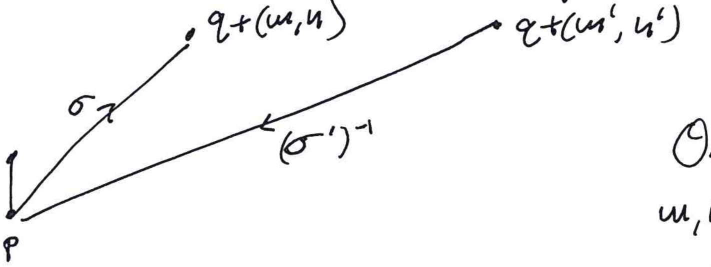
We want to estimate $\mu(A \triangle A')$ where A corresponds to $\sigma = \sigma(m,n)$ and A' corresponds to $\sigma' = \sigma'(m', n')$.

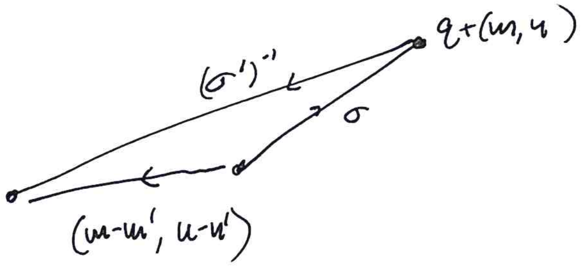
Now ι interchanges A and B. It interchanges A' and B' so it leaves invariant $A \triangle A'$. Thus $A \triangle A'$ is $\pi^{-1}(X)$ where $X \subset T$. The boundary of $A \triangle A'$ is the union of the boundary of A (which is $\sigma_0 \cup \sigma_1$) and the boundary of A' (which is $\sigma_0' \cup \sigma_1'$). The boundary of X is the projection of the boundary of $A \triangle A'$ so it is $\sigma \cup \sigma'$.

The geometry of $A \triangle A'$ on T.

σ and σ' both travel from p to q. The curve $\sigma \cdot (\sigma')^{-1}$ represents a closed curve on T.

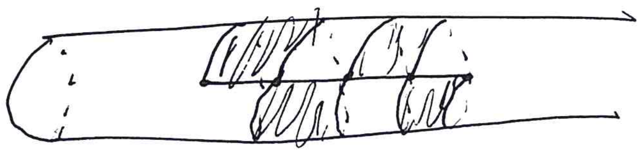
This closed curve represents the homology class $(m-m', n-n')$ on T.





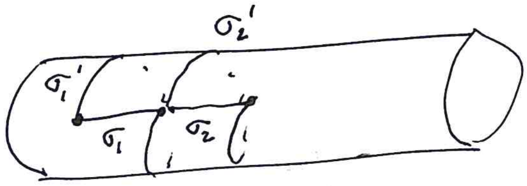
Our assumption that m, m' are even and n, n' are odd means that this vector is not primitive, ie its coordinates are not relatively prime. ~~So~~ In particular they are both even. Thus $\sigma \cdot (\sigma')^{-1}$ cannot be represented by a simple curve, it must have self-intersections. If $(m-m', n-n') = d(\bar{m}, \bar{n})$ where $\bar{m}$ and $\bar{n}$ are relatively prime then $\sigma \cdot (\sigma')^{-1}$ has $d-1$ self-intersections which divide σ and σ' into d segments.

Picture



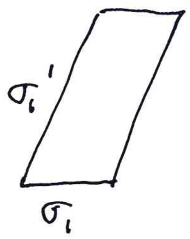
The curve ~~as~~ $d(\bar{m}, \bar{n})$ is trivial in $\mathbb{Z}_2$ homology so is a boundary. We can construct the surface whose boundary is this curve.

Let $v_1 \ldots v_{d-1}$ be the intersection points of σ and σ'. The points $p, v_1, \ldots v_{d-1}, q$ divide σ into intervals $\sigma_1 \ldots \sigma_d$ and σ' into intervals $\sigma_1' \ldots \sigma_d'$. The intervals σ_1, σ_2 and σ_1', σ_2' form the boundary of a parallelogram. Let $|u \times v|$ denote the area of the parallelogram spanned by u and v.



For saddle connections $|\sigma \times \tau| = |hol(\sigma) \times hol(\tau)|$.

The surface consists of $d/2$ ~~reda~~ parallelograms each with area



$$|\sigma_i \times \sigma_i'| = \left|\frac{\sigma}{d} \times \frac{\sigma'}{d}\right| = \frac{1}{d^2}|\sigma \times \sigma'|.$$

The total area is $\frac{1}{2d}|\sigma \times \sigma'|$. The area of $A \Delta A'$ is twice this. So an upperbound is $|\sigma \times \sigma'|$.

If $\sigma = \sigma(m,n)$ and $\sigma' = \sigma(m', n')$ then

$hol(\sigma) = (m, n+1-2s)$ $\quad hol(\sigma') = (m', n'+1-2s) = (2a, 2b+1+1-2s)$

insures our parity condition.

$$|\sigma \times \sigma'| = \left|\det\begin{pmatrix} m & m' \\ n+1-2s & n'+1-2s \end{pmatrix}\right|$$

$$= \left|\det\begin{pmatrix} m & 2a \\ n+1-2s & 2b+2-2s \end{pmatrix}\right| = m(2b+2-2s) - 2a(n+1-2s)$$

$$= 2mb + m(2-2s) - 2a(n+1-2s)$$

$$= 2(n+1-2s)\left(\underbrace{\frac{2m}{n+1-2s}}_{A} \cdot b - a + \underbrace{\frac{m(2-2}{2(n+1-}}_{B}\right.$$

$(\alpha b - a + \beta)$ with α irrational

can be made as small as we like since $b\alpha$ is dense mod 1. Solve $b\alpha - \beta \bmod 1 < \varepsilon$ then choose a so that $|\alpha b - a + \beta| < \varepsilon$.

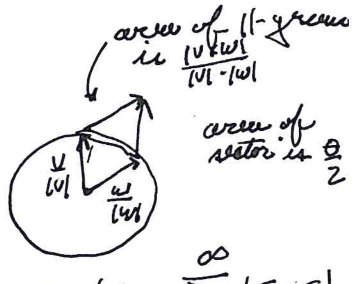
Remaining conditions
If θ_j angle of σ_j then $\theta_j \to \theta_\infty$.
h_j = perp proj. to θ_∞
$h_j \to 0$.

Using this we can find sequences σ_j so that $\sum_{j=1}^{\infty} |\sigma_j \times \sigma_{j+1}| < \infty$.

Let's assume that $|\sigma_j|$ is increasing. (always true for a subsequence)

Now if the angle θ between v and w is less than $\frac{\pi}{2}$ then

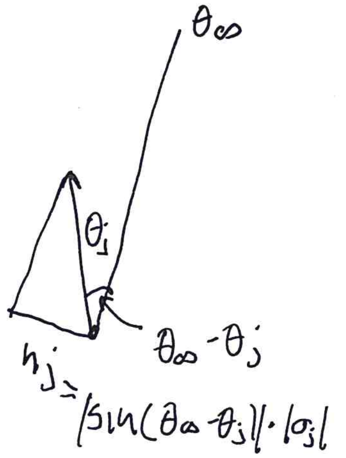
$$\frac{\theta}{2} \le \frac{|v \times w|}{|v| \cdot |w|} \le \theta.$$



If θ_j is the angle of σ_j then $|\theta_i - \theta_j| \le \sum_{k=i}^{j} |\theta_i - \theta_{i+1}| \le c \sum_{k=i}^{\infty} |\sigma_i \times \sigma_j|$

since $|\sigma_i|$ is bdd below

So θ_j form a Cauchy sequence and $\theta_j \to \theta_\infty$.



$$h_j \approx |\sigma_j| \cdot |\sin(\theta_\infty - \theta_j)|$$
$$\le |\sigma_j| \cdot |\theta_\infty - \theta_j|$$
$$\le |\sigma_j| \cdot \sum_{i=j}^{\infty} \frac{|\sigma_i \wedge \sigma_{i+1}|}{|\sigma_i| \cdot |\sigma_{i+1}|}$$
$$\le |\sigma_j| \cdot \sum_{i=j}^{\infty} \frac{|\sigma_i \wedge \sigma_{i+1}|}{|\sigma_j|^2}$$
$$\le \frac{c}{|\sigma_j|} \sum_{i=j}^{\infty} |\sigma_i \wedge \sigma_{i+1}|$$

Note
So in fact $h_j \cdot |\sigma_j|$ goes to 0.
Interesting in terms of renormalization picture.

This is all we need to construct a single non-ergodic direction. To construct uncountably many we need to organize this procedure so that we choose two successors to each (m,n). Furthermore we need to assure that each ~~a~~ branch of our tree corresponds to ~~the~~ a distinct direction. We can do this by ~~checking that~~ choosing directions of offspring close to directions of parents.

Cor. There are minimal non-ergodic directions. Dense traj that spend more than 1/2 their time on the r.h.s of the table.

Comments. If s is rational then $|V(m,n) \wedge V(m',n')| \geq \varepsilon > 0$ for some ε depending on s.

In this case every direction is either totally periodic or uniquely ergodic.

If s is Diophantine then the set of ~~directions~~ non-ergodic directions has Hausdorff dimension $1/2$. (Cheung).

Thm. ~~A~~ (Masur-S). For every stratum $\mathcal{H}(\alpha_1 \dots \alpha_n)$ other than $\mathcal{H}(0)$ there is a constant $c > 0$ so that for almost every surface in that stratum the Hausdorff dimension of non-ergodic directions is c

Thm. ~~Cheung~~ (Masur) $c \leq 1/2$.

Thm. (Athreya-Chaika) For $\mathcal{H}(2)$ $c = 1/2$.

~~the~~

Def. Let S, S' be translation surfaces. Let $f: S \to S'$. f is affine if it is continuous, ~~differentiable as a~~ takes singular points Σ to singular points Σ', is differentiable on $S - \Sigma$ and has constant derivative.

~~If $Df \neq I$~~

If S, S' are half-translation surfaces then Df is defined ~~as an element of $PGL(2,\mathbb{R})$. In this case we~~ modulo $\pm I$. We say f is affine if Df is constant ~~as a~~ in $GL(2,\mathbb{R})/\pm I$.

Def. ~~Call~~ An affine map f is a translation ~~surface~~ equivalence if $Df = I$ ($Df = \pm I$ in half-trans. case).

Def. If $S = S'$ then an affine map is an affine automorphism. Note: ~~det~~ In this case $\det Df = \pm 1$.

An affine automorphism of S is elliptic if both eigenvalues of Df are on the unit circle but not I.

It is parabolic if Df has eigenvalues ± 1 but $Df \neq id$.

It is ψ-Anosov if one eigenvalue of Df is outside the unit circle and one is inside.

Equivalently f is ψ-Anosov if it has 2 invariant line fields one expanded by $\lambda > 1$ and the other contracted by ~~$\frac{1}{\lambda}$~~ a factor of $1/\lambda$

The property of being pseudo-Anosov has some purely topological consequences.

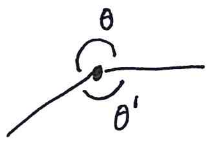
↗ use a ½ trans surface, no cone angle π sing. pts.

Theorem. If $f: S \to S$ is pseudo-Anosov and γ is a simple closed curve on S then $f^n\gamma$ is not <u>isotopic</u> to γ unless $n = 0$.

Assume trans. surf. rather than ½ trans.

Proof. Say that $Df = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $|\lambda| > 1$.

γ has a representative $\bar{\gamma}$ of minimal length which is unique. (Cat(0) condition) This representative consists of a finite number of straight line segments. with endpoints at singular points.



$\theta, \theta' \geq \pi$.

$f^n(\bar{\gamma})$ is the unique representative for $f^n(\gamma)$ of minimal length.

In $f^n(\bar{\gamma})$ the horizontal coordinate of each segment is multiplied by λ^n so $f^n(\bar{\gamma}) \neq \bar{\gamma}$ so $f^n(\gamma)$ is not homotopic to γ unless $\lambda^n = 1$ or $n = 0$.