MA475 Example Sheet 3

12 February 2019

1. Let f and g be nonconstant maps defined in connected domains in $\mathbb{C}$. Show the chain rule for valence: $v_{fg}(w) = v_f(g(w)) \cdot v_g(w)$.

2. Let $D = \{z : 0 < |z| < 1\}$. Let G be the group generated by $z \mapsto z\exp(2\pi i/n)$ for some fixed n. Identify D/G up to conformal equivalence.

3. Let $D = \mathbb{C} - \{0\}$ and let G be generated by $g : z \mapsto z/2$. Show that D/G is a torus.

4. Show that the surfaces $\mathbb{C}_\infty$, S and $\mathbb{CP}^1$ are conformally equivalent.

5. Give $\mathbb{C}$ two atlases $\mathcal{A}_1$ consisting only of the identity map and $\mathcal{A}_2$ consisting of the map $h(z) = \bar{z}$. Show that $\mathcal{A}_1$ and $\mathcal{A}_2$ determine different conformal structures on $\mathbb{C}$ but that $h : (\mathbb{C}, \mathcal{A}_1) \to (\mathbb{C}, \mathcal{A}_2)$ is a holomorphic equivalence.

6. Let S be the metric space defined by $S = \{(z, w) \in \mathbb{C}_\infty \times \mathbb{C}_\infty : z = w^n\}$. Show that S is a surface and that $S \cup \{(\infty, \infty)\}$ is homeomorphic to $\mathbb{C}_\infty$.

7. Derive the identity

$$\left(\frac{\pi}{\sin \pi z}\right)^2 = \sum_{n=-\infty}^{+\infty} \frac{1}{(z-n)^2}$$

as follows.

   (a) Show that the difference $d(z)$ between the two sides is holomorphic in $\mathbb{C}$ and satisfies $d(z+1) = d(z)$.

   (b) Show that $d(x + iy) \to 0$ as $|y| \to +\infty$ uniformly in x.

   (c) Deduce that d is constant and hence zero in $\mathbb{C}$.