MA475 Example Sheet 1

16 January 2019

1. Recall that $\omega_2 = \frac{-ydx+xdy}{x^2+y^2}$. Compute that

$$d(\arctan(\frac{y}{x})) = \frac{-ydx + xdy}{x^2 + y^2}.$$

The first expression gives the angle θ in polar coordinates when $x > 0$.

2. Compute the complex valued 1-form $exp^*(\frac{dz}{z})$.

3. Say we have an l-form θ and an m-form ψ show that $\theta \wedge \psi = (-1)^{lm} \psi \wedge \theta$.

4. Recall that we defined the pullback of a k-form θ by a smooth function F to be F^* where this form is defined on a k-tuple of vectors by $F^*(\theta)(v_1 \dots v_k)$ to be $\theta(DF(v_1) \dots \theta(DF(v_k))$. Show that for a 1-form θ that $(F \circ G)^*(\theta) = G^*(F^*(\theta))$.

5. Let θ be a 1-form. Show that $\int_\gamma F^*(\theta) = \int_{F \circ \gamma} \theta$.

6. A consequence of Cauchy's theorem is that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{n+1}} dw.$$

Deduce from this that power series expressions for holomorphic functions can be differentiated term by term.

7. As sequence if functions f_n converges locally uniformly to f in a domain D if $f_n \to f$ on each compact subset of D. Morera's theorem shows that if the holomorphic functions f_n converge locally uniformly then the limit is holomorphic.

Show that the the series

$$\sum_{|n|>N} \left(\frac{1}{z-n} + \frac{1}{n} \right)$$

converges uniformly in the disc $\{z : |z| < N\}$.

Deduce that

$$\frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right)$$

converges to a holomorphic function in $\mathbb{C} - \mathbb{Z}$. Find the derivative of this function.