Thm. Let f be a holomorphic function on an open nbd U of 0 in $\mathbb{C}$ with $f(0)=0$. Suppose $f'(0)\neq 0$ then there is a nbd $U'\subset U$ of 0 such that f is a homeomorphism onto its image $f(U')\subset\mathbb{C}$ and the inverse to $f|U'$ is holomorphic.

Proof. (This is the inverse fcn. theorem. We approach it in the same spirit as the implicit fcn. thm.)

Since 0's of non-constant functions are isolated there is a disk ~~$\Delta \subset U$~~ Δ with $0\in\Delta\subset U$ where $f(z)\neq 0$ for $z\in\Delta-\{0\}$.

Now $\frac{1}{2\pi i}\int_{\partial D}\frac{f'(z)}{f(z)}\,dz$ counts the # of solutions of $f(z)=0$ in Δ with mult.

Since $f'(0)\neq 0$ there is 1 soln of mult. 1

so $\frac{1}{2\pi i}\int_{\partial D}\frac{f'(z)}{f(z)}\,dz = \underline{1}$.

$\mu(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)-w}\, dz$ counts the # of solns of

(Replace $f(z)$ by $g(z) = f(z)-w$. note that the $g'(z) = f'(z)$)

$f(z) = w$ in D.

$g(z) = 0 \Leftrightarrow f(z) = w$

By compactness $|f(z)| \geq \varepsilon$ on

∂W so for $|w| < \varepsilon$ $\mu(w)$ is continuous and

$\mu(w) = \mu(0) = 1$.

Let $W' = f^{-1}(\underbrace{\{|z| < \varepsilon\}}_{= \Delta_\varepsilon})$.

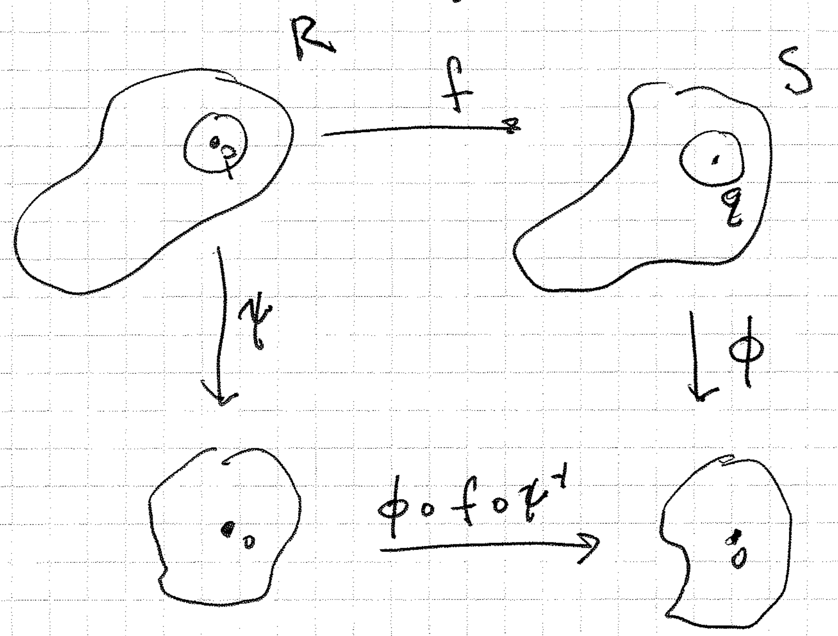
Let $\phi(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{z f'(z)}{f(z)-w}\, dz$. When there is

a unique soln to $f(z) = w$ it is given by

$\phi(w)$. (Replace $f(z)$ by $g(z) = f(z)-w$

$\phi(w) : \Delta_\varepsilon \to W'$. As before ϕ is holomorphic.

Remark. This result can be restated for Riemann surfaces.



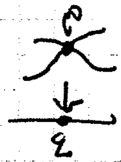
Can say $f'(p) \neq 0$ even if we can't identify $f'(p)$.

Order of f at p is the order of the zero of $\phi \circ f \circ \psi^{-1}$ at 0.

If f has order 1 then f is a local homeomorphism with hol. inverse.

Order 1 means $f'(p) \neq 0$ in any coord charts.

Lemma. Let f be a holomorphic function on an open nbd. U of 0 in $\mathbb{C}$ with $f(0)=0$ but f not identically 0. Then there is a unique integer $k \geq 1$ such that on some smaller nbd U' of 0 we can find a holomorphic function g with $g'(0) \neq 0$ and $\underline{f(z) = g(z)^k \text{ on } U'}$.



Proof. $f(z) = a_k z^k + a_{k+1} z^{k+1} + \dots \qquad a_k \neq 0$

$$f(z) = a_k z^k (1 + b_1 z + b_2 z^2 + \dots) \quad \text{where } b_j = \frac{a_{k+j}}{a_k}.$$

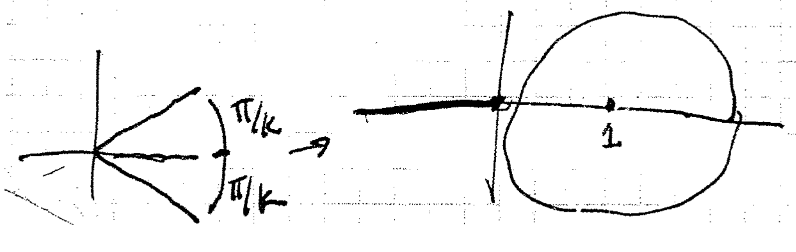
Assume $0 \in U' \subset U$ is small enough that

$|\sum b_j z^j| < 1$ then the image of

$z \mapsto (1 + b_1 z + b_2 z^2 + \dots)$ is contained

in a disk in $\mathbb{C} - \{0\}$. In particular we can choose a branch of $z \mapsto z^{1/k}$ in this disk and

define $h(z) = (1 + b_1 z + b_2 z^2 + \dots)^{1/k}$



Let $g(z) = a_k^{1/k} z\, h(z)$ for some choice of k-th root

then $g^k(z) = a_k z^k (1 + b_1 z + b_2 z + \dots) = f(z)$.

Furthermore since $g'(0) = a_k^{1/k}$
we have that g is locally invertible with a holomorphic inverse

Terminology. If $f: U \xrightarrow{C^{\mathbb{C}}} \mathbb{C}$ and $f(z_0)=0$

~~then there~~ and $f(z) = \sum_{j=k}^{\infty} c_j z^j$ with $c_k \neq 0$ then

k is the order of vanishing of f at z_0. $\mathrm{ord}_{z_0}(f)$.

(If f is locally constant we set the order of vanishing equal to ∞). $\boxed{\mathrm{ord}_{z_0}(f) = \infty}$

We saw last time that if f has order of vanishing $k^{<\infty}$ then f ~~has~~ a local k-th root, function g with $g'(z_0) \neq 0$.

$\mathrm{ord}(f)$ does not depend on the choice of coordinates in the domain

If $f(z_0) = w_0$ then we define the local multiplicity of f at z_0 to be the order of vanishing of $z \mapsto f(z) - w_0$. We write this as $V_p(f)$.

$f: R \to \mathbb{C}$ $\mathrm{ord}_{p}(f)$ makes sense

Thus $f(z) = w_0 + \sum_{j = V_p(f)} a_j z^j$.

The local multiplicity is independent of the coordinate system in the domain and range in the range so it makes sense ~~even~~ for maps between Riemann surfaces whereas the order of vanishing makes sense for maps from Riemann surfaces to $\mathbb{C}$ specifically.

in particular it does not depend on the value of the image point being 0.

$f: R \to S$, $V_p(f)$ makes sense.

Theorem. (Local model for holomorphic maps).

Let $f: R \to S$ be holomorphic with

$f(p) = q$ and $n = \nu_p(f) < \infty$.

Let V be a nbd of q, Let $\psi : V' \to \mathbb{C}$ be a chart

with $\psi(q) = 0$.

Then there there is a nbd U of p and a chart $\phi : U \to \mathbb{C}$ with $\phi(p) = 0$ and

$(\psi \circ f)(z) = (\phi(z))^n$.

$$\begin{tikzcd}
p \in U \subset R \arrow[r, "f"] \arrow[d, "\phi"] & q \in V \subset S \arrow[d, "\psi"] \\
0 \in D(0,\varepsilon) \subset \mathbb{C} \arrow[r, "z \mapsto z^n"] & 0 \in D(0,\varepsilon') \subset \mathbb{C}
\end{tikzcd}$$

Proof of theorem. We are given V and (psi) ψ. Choose a W nbd. U' of p so that $f(U') \subset V$.

$$\begin{tikzcd}
p \in U' \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, "\psi \circ f"] & q \in V \arrow[d, "\psi"] \\
\mathbb{C} \arrow[r, "z \mapsto z^n"] & \mathbb{C}
\end{tikzcd}$$

Note ψ is part of our initial data. We construct ϕ.

Apply our lemma to the maps $\psi \circ f$ to give us a g

$$\begin{tikzcd}
& \cdot \arrow[dr, "z \mapsto z^n"] & \\
\cdot \arrow[ur, "g"] \arrow[rr, "\psi \circ f"] & & \cdot
\end{tikzcd}$$

Last step is to observe that $g'(p) \neq 0$ so g is a local homeomorphism on some set $U \subset U'$. Let us ~~call g|U~~ rename $g|U$ as ϕ so we get:

our lemma from last time gives using the inverse function theorem

$$\begin{tikzcd}
\cdot p \in U \arrow[r, "f"] \arrow[d, "\phi"] & \cdot q \in V \arrow[d, "\psi"] \\
\cdot \arrow[r, "z \mapsto z^n"] & \cdot
\end{tikzcd}$$

as was to be shown.

Corollary. If a holomorphic map is locally injective then it is a local homeomorphism.

Proof. If $n > 1$ then $z \mapsto z^n$ is not locally injective so ~~must~~ $V_p(f) = 1$ at every

$\Rightarrow f'(p)$

$\Rightarrow f$ is a locally conform

Corollary. If a holomorphic map is a bijection then it is a conformal equivalence.

Proof. If $f : R \to S$ is a bijection then there is ~~an~~ a function $g = f^{-1} : S \to R$. g is locally conformal so g is conformal.

Proper holomorphic maps

Definition. A continuous function $f: R \to S$ is proper if the inverse image of a compact set is compact. ~~Recall that the image of a compact set~~ is compact.

Example: A finite sheeted covering map is proper.

Non-example: The inclusion of the open disk into $\mathbb{C}$ is not proper. (Example sheet:

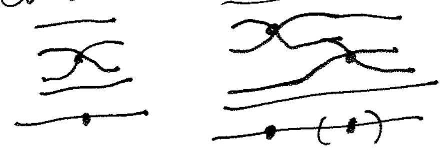
Assume from now on that all Riemann surfaces are Hausdorff. makes an appearance in locally non-constant

Theorem. Let $f: R \to S$ be a proper holomorphic map between Hausdorff Riemann surfaces then for each point $q \in S$ there is a nbd. U_q of q so that $f^{-1}(U)$ consists of finitely many open sets $U_1 \dots U_k$ and $f|U_j \to U_q$ is conjugate to $z \mapsto z^{n_j}$

for $n_j = \nu_{p_j}(f)$.

Note that $n_j = \nu_{p_j}(f)$ according to our notation.

Remember $n_j = 1$ is the typical case

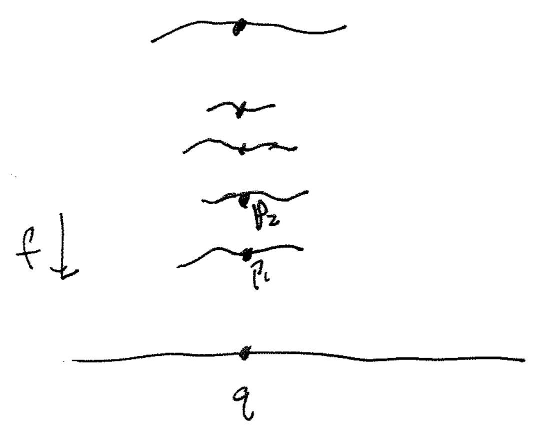


We can choose a coordinate chart around q and around each p_j so that

$$\begin{tikzcd} p_j \subset R \arrow[r, "f"] \arrow[d, "\phi_{p_j}"] & q \subset S \arrow[d, "\psi_q"] \\ D(0,\varepsilon) \subset \mathbb{C} \arrow[r, "z \mapsto z^n"] & D(0,\varepsilon^n) \subset \mathbb{C} \end{tikzcd}$$

$$\psi_q \circ f \circ \phi_{p_j}^{-1}(z) = z^{n_j}.$$

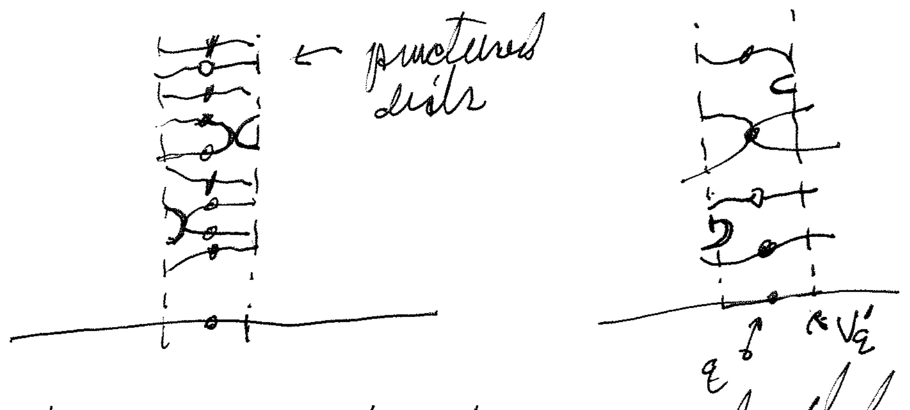
Step 1.



Since $f^{-1}(q)$ is finite, $\bigcap_j f(U_j)$ is a nbhd of q.

Call it V_q. If we replace U_j by $U_j' = U_j \cap f^{-1}(V_q')$

then we get this picture



Want $V_q' = \psi_q^{-1}(D(0,\varepsilon))$.

$f|U_j'$ has the prescribed form.

The problem is that $f^{-1}(V_q')$ may contain other points as well as those in some U_j'. (If not we are done.)

$\boxed{\text{i.e. there may be components of } f^{-1}(V_q') \text{ that don't meet } f^{-1}(q).}$

Consider $\bar{V}_q = \psi_q^{-1}(\bar{D}(0,\varepsilon))$.

$f^{-1}(\bar{V}_q)$ is compact by the properness assumption

so $f^{-1}(\bar{V}_q) - \bigcup_j U_j'$ is compact ~~by~~ since U_j' is open so $f^{-1}(\bar{V}_q) - \bigcup_j U_j$ is a closed subset of a Hausdorf space and hence compact, $f(f^{-1}(\bar{V}_q) - \bigcup_j U_j)$ is compact and does not contain q. Now a compact space is closed ~~so~~ (Hausdorfness) so there is some nbd. $V_q'' = \psi_q^{-1}(D(0,\varepsilon''))$ disjoint from $f(f^{-1}(\bar{V}_q) - \bigcup_j U_j)$. In particular $f^{-1}(V_q'')$ is disjoint from $f^{-1}(\bar{V}_q) - \bigcup_j U_j$

Since $f^{-1}(V_q'')$ is contained in $f^{-1}(\bar{V}_q)$ but disjoint from $f^{-1}(\bar{V}_q) - \bigcup_j U_j$ it must be contained in $\bigcup_j U_j$. This is what we wanted to show.

Completion of the proof from last time.

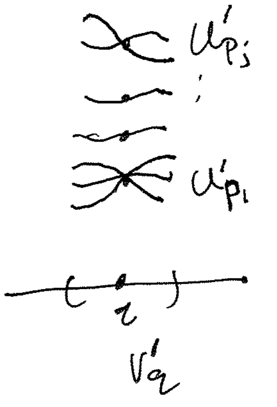
$f : R \to S$ is holomorphic and proper and $\nu_p(f) < \infty$ for each $p \in R$.

Want to show that
for each $q \in S$ there is a nbd. V_q, coordinate chart $\psi : V_q \to D(0, \varepsilon')$ and nbds U_{p_j} for $p_j \in f^{-1}(q)$ and coordinate charts $\phi_j : U_{p_j} \to D(0, \varepsilon)$ so that

$$\begin{array}{ccc} U_{p_j} & \xrightarrow{f} & V_q \\ \downarrow \phi_j & & \downarrow \psi \\ D(0,\varepsilon_j) & \xrightarrow{z \mapsto z^n} & D(0,\varepsilon_j'). \end{array}$$

(This is a little more precise than the statement of the thm.)

Have $f^{-1}(q)$ is finite. We have the local picture on a nbd of $f^{-1}(q)$.

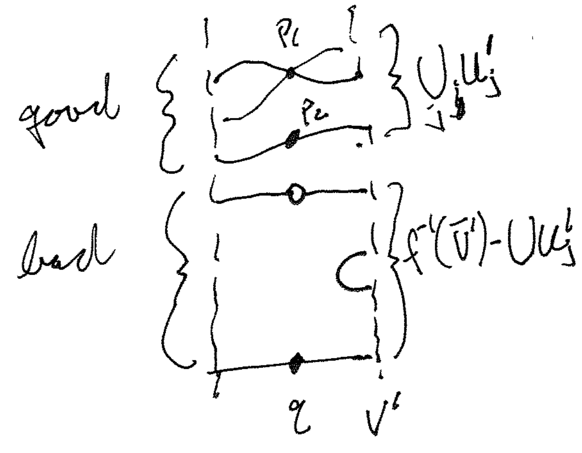


Need to show that we have a $V''_q \subset V'_q$ so that if $U''_{p_j} = U'_{p_j} \cap f^{-1}(V''_q)$ then

$$f^{-1}(V''_q) = \bigcup_j U''_{p_j}.$$

$U'_{p_j} \subset U_{p_j}$. May assume that $U'_{p_j} \subset \overline{U'_{p_j}} \subset U_{p_j}$

There are two potential obstructions:



i.e. two types of potential components of $f^{-1}(V_q')$ that don't meet $f^{-1}(q)$.

The first are ruled out by properness. We deal with the second by choosing a smaller nbd. $V'' \subset V'$. where $V'' = \psi^{-1}(D(0,\varepsilon''))$ $\varepsilon''<\varepsilon'$.

~~assume~~ Let $\bar V'$ be the closure of V' in S. We may assume that $\bar V'$ is compact. Thus $f^{-1}(\bar V')$ is compact by properness.

Now $\bigcup_j U_j'$ is open in R so $f^{-1}(\bar V') - \bigcup_j U_j'$ is a closed subset of a compact set so it is compact (using Hausdorfness of R).

Now $f\left(f^{-1}(\bar V') - \bigcup U_j'\right)$ is compact, hence closed and does not contain q. So there is a nbd V_q'' of q which is not in $f(f^{-1}(\bar V') - \bigcup U_j')$.

Thus $f^{-1}(V_q'')$ is disjoint from $f^{-1}(\bar V') - \bigcup U_j'$.

Since $f^{-1}(V_q'')$ is contained in $f^{-1}(\bar{V}')$ but disjoint from $f^{-1}(\bar{V}') - \bigcup_j^{\circ} \mathcal{U}_j$ it must be contained in $\bigcup_j \mathcal{U}_j'$.

Now take as nbds of the p_j, $\mathcal{U}_j'' = \mathcal{U}_j' \cap f^{-1}(V_q'')$ and we are done.

Remark. The same proof can be used to show the strictly topological fact that:

Prop. A proper local homeomorphism is a covering map.

In order to apply our theorem in the holomorphic setting we need to resolve a technical point relating the local condition $\nu_p(f) < \infty$ to the global condition of not being constant.

Prop. Let $f: R \to S$ be a non-constant holomorphic function between Riemann surfaces where R is connected.

If f is not constant then the sets

$f^{-1}(q)$ consist of isolated points (with $\nu_p(f) < \infty$).

Proof. The set $f^{-1}(q)$ is closed.

The union of isolated points in $f^{-1}(q)$ is open. Thus the set of of non-isolated points in $f^{-1}(q)$ is closed. On the other hand the set of non-isolated points in $f^{-1}(q)$ is open. Let $p_j \to p_\infty$ be points in $f^{-1}(q)$. By choosing coordinate charts we can reduce to the case where $p_j \in U \subset \mathbb{C}$ and $f(p_j) \in \mathbb{C}$. Now

$z \mapsto f(z) - f(q)$ is holomorphic and has a convergent sequence of 0's.

$\cdot \; \cdots \, p_\infty$ It follows that this function vanishes on a nbd. of p_∞.

(Follows from complex analysis course.)

Thus the set of non-isolated points in $f^{-1}(q)$ is open and closed. If it is non-empty then it is all of R and f is constant. Conclude that it is empty.

Theorem. Let $f: R \to S$ be a proper holomorphic non-constant map between connected Riemann surfaces. Then the quantity $\sum_{p \in f^{-1}(q)} V_p(f)$ is constant (and finite) and we call it the degree of f.

Proof. It suffices to show that the quantity is locally constant. Given $q \in S$ choose a nbd. V_q as in the previous theorem. Let $\psi: V_q \to D(0,\varepsilon')$ be a chart coord. chart and $\Phi_j: U_{p_j} \to D(0,\varepsilon)$ ~~bar~~ be as before.

Let $z \in D(0,\varepsilon')$. ~~I~~ Let's consider the set

$$f^{-1} \circ \psi^{-1}(z) = \bigcup_j f^{-1} \circ \psi^{-1}(z) \cap U_j$$

For a fixed j we can apply Φ_j to $f^{-1} \circ \psi^{-1}(z) \cap U_j$

$$\begin{array}{ccc} U_j & \xrightarrow{f} & V_q \\ \downarrow \Phi_j & & \downarrow \psi \\ D(0,\varepsilon) & \xrightarrow{z \mapsto z^n} & D(0,\varepsilon') \end{array}$$

If $z \neq 0$ then $\phi_j(f^{-1}\circ\psi^{-1}(z) \cap U_j)$ consists of

$n = n_j = \nu_{p_j}(f)$ points each with $\nu(f) = 1$.

If $z = 0$ then we get a single point (q) with

local degree $\nu_{p_j}(f)$. Thus for each j the contribution

is independent of $z \in D(0, \varepsilon')$.

The total degree is the sum of ~~degrees~~ contributions for the different j's.

Our plan is to use these tools to investigate meromorphic functions. This will turn out to be useful tools for understanding Riemann surfaces.

First result:

Thm. Let f be a meromorphic function on a compact connected Riemann surface R. Then f has the same number of zeros as poles counted with multiplicity.

Proof. A meromorphic function on R is a holomorphic function to $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. According to the theorem $\sum_{p \in f^{-1}(0)} \nu_p(f) = \sum_{p \in f^{-1}(\infty)} \nu_p(f)$.

The first quantity is the number of zeros counted with multiplicity. The second quantity is the number of poles counted with multiplicity