Definition. A surface S is a topological space together with a family of maps

$\mathcal{A} = \{\phi_\alpha : \alpha \in \mathcal{A}'\}$ $\qquad$ $\phi_\alpha : U_\alpha \overset{\subset S}{\longrightarrow} W_\alpha \subset \mathbb{C}$

atlas $\qquad$ index set $\qquad$ charts

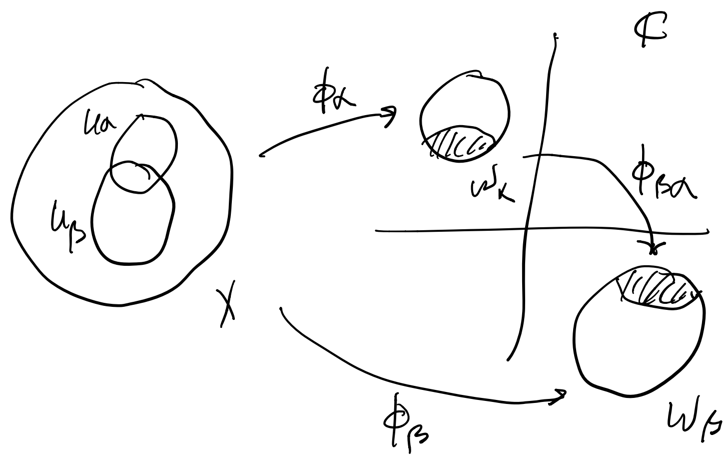
so that

(1) Each ϕ_α is a homeomorphism of an open set onto an open set $W_\alpha \subset \mathbb{C}$.

(2) $\{U_\alpha : \alpha \in \mathcal{A}'\}$ is a cover of S.

If U_α meets U_β then $\phi_{\beta\alpha} = \phi_\beta \circ \phi_\alpha^{-1}$ is a homeomorphism of $\phi_\alpha(U_\alpha \cap U_\beta)$ to $\phi_\beta(U_\alpha \cap U_\beta)$.

The $\phi_{\beta\alpha}$ are called transition functions.



In this diagram we have $\phi_{\beta\alpha} \circ \phi_\alpha = (\phi_\beta \circ \phi_\alpha^{-1}) \circ \phi_\alpha = \phi_\beta$.

There are two technical issues which I will defer. Is X 2nd countable? Hausdorff?

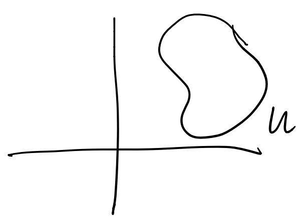
Definition. A surface is a Riemann surface if the transition functions $\phi_{\alpha\beta}$ are holomorphic. (These map open sets in $\mathbb{C}$ to open sets in $\mathbb{C}$.)

Remark. $\phi_{\beta\alpha} = \phi_\beta \circ \phi_\alpha^{-1}$ and $\phi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ so

$\phi_{\beta\alpha} \circ \phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} \circ \phi_\alpha \circ \phi_\beta^{-1} = I_d$. It follows that

$\phi_{\beta\alpha}$ has a holomorphic inverse so $\phi_{\beta\alpha}$ is conformal. We can call $\mathcal{A}$ a holomorphic or conformal atlas.

Example 1. ("Trivial example"). U is an open set in $\mathbb{C}$.



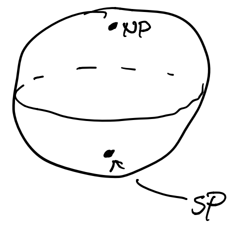
Take $\mathcal{A} = \{\phi_0\}$ where $\phi_0 : U \to \mathbb{C}$ is the inclusion.

$U_0 = W_0 = U$.

Example 2. The round sphere.

Let $S^2 = \{(x, y, t) : x^2 + y^2 + t^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$.

Let $NP = (0, 0, 1)$ and $SP = (0, 0, -1)$.

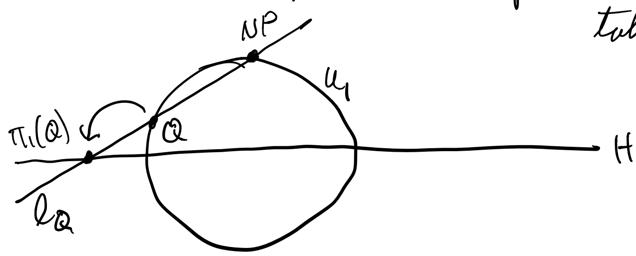


Let $U_1 = S^2 - NP$, $U_2 = S^2 - SP$

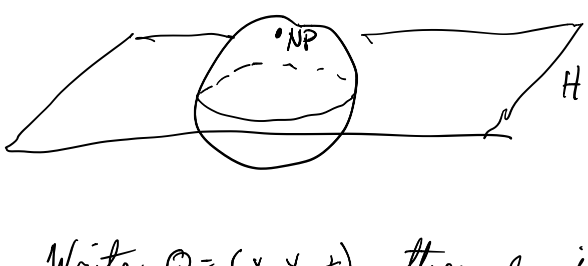
Let $H \subset \mathbb{R}^3 = \{(x, y, 0)\}$.

Let $\alpha_1 : H \longrightarrow \mathbb{C}$ be $\alpha_1((x, y, 0)) = x + iy$.

Consider a projection map π_1 from U_1 to H which takes a point Q to the intersection of $\overline{NP, Q}$ with H.



Define $\phi_1 : U_1 \longrightarrow \mathbb{C}$ to be $\alpha_1 \circ \pi_1$.



Write $Q = (x, y, t)$ then l_Q is given by

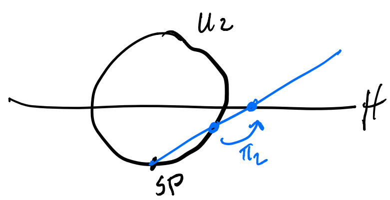
$$s \hookrightarrow (1-s)(0,0,1) + s(x, y, t) = (sx, sy, (1-s) + st)$$

To find the point of intersection between l_Q and H we set $1-s+st=0$ to get $1=s-st$

$1=s(1-t)$ or $s=\frac{1}{1-t}$. Thus $\pi_1(Q)=\left(\frac{x}{1-t},\frac{y}{1-t},0\right)$

and $\phi_1(Q)=\alpha_1\circ\pi_1(Q)=\frac{x+iy}{1-t}$.

We define $\phi_2 : U_2 \to \mathbb{C}$ to be $\alpha_2\circ\pi_2$ where π_2 is the projection from SP and $\alpha_2 : (x,y,0)\to x-iy$.



Note the minus sign.

$s \mapsto (sx, sy, (1-s)-st)$

$(1-s)-st=0$ $\quad$ $1=s(1+t)$

$s=\frac{1}{1+t}$.

Calculating as before

$\pi_2((x,y,t))=\left(\frac{x}{1+t},\frac{y}{1+t},0\right)$

$\phi_2=\alpha_2\circ\pi_2=\frac{x-iy}{1+t}$.

Remark. The 2-sphere has a well defined orientation. $\mathbb{C}$ has a well defined orientation.

Let $Q\in S^2$. Let N_Q be the outward pointing normal vector at Q. Let v,w be an orthogonal basis of tangent vectors to S^2 at Q. We say that $\langle v,w\rangle$ is a positive basis if

$N_Q \times V = V$ (?). ϕ_1 is orientation preserving. We modified ϕ_2 so that ϕ_2 was also orientation preserving.

We want to compute the transition map ϕ_{21}. We do this indirectly.

Observe that for $Q \in U_1 \cap U_2$ we have

$$\phi_1(Q) \cdot \phi_2(Q) = \frac{x+iy}{1-t} \cdot \frac{x-iy}{1+t} = \frac{x^2+y^2}{1-t^2} = 1 \in \mathbb{C}.$$

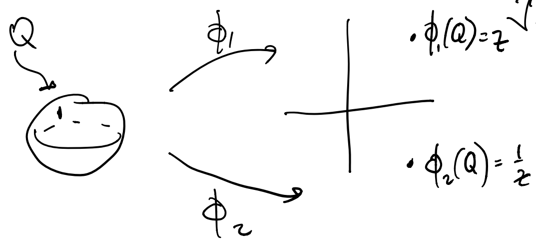
complex mult.

Recall that $x^2+y^2+t^2=1$, $x^2+y^2=1-t^2$

So $\phi_1(Q) = \dfrac{1}{\phi_2(Q)}$.

Set $z=\phi_1(Q)$.

Then $\phi_2(Q) = \frac{1}{z}$



$\phi_{21}(\phi_1(Q)) = \phi_2(Q)$ by def. of ϕ_{21}. This gives $\phi_{21}(z) = \frac{1}{z}$. This function is indeed holomorphic where it is defined (on $\phi_1(U_1 \cap U_2) = \mathbb{C} - \{0\} = \phi_2(U_1 \cap U_2)$).

Thus $\mathcal{A}$ is a holomorphic atlas.

Let's examine the logic of this construction.

The charts ϕ_1, ϕ_2 are homeomorphisms.
We are not saying that they are "holomorphic".

What would holomorphic mean in this setting?

The holomorphic maps are the transition maps. These map between open subsets of $\mathbb{C}$.

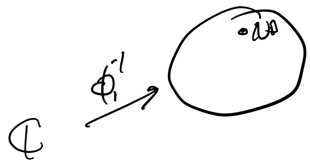
In other words the only role played by $S^2 \subset \mathbb{R}^3$ is to be a topological space.

Recall that $\mathbb{C}_\infty$ ($= \mathbb{C} \cup \{\infty\}$) is the one-point compactification of $\mathbb{C}$. The topology is defined so that an open set $U \subset \mathbb{C}$ is a nbd. of ∞ in $\mathbb{C}_\infty$ if the complement of U is compact.

Equivalently a sequence of points converges to ∞ if and only if it eventually leaves all compact subsets.

We can identify S^2 with $\mathbb{C}_\infty$ as follows.

Consider $\phi_1^{-1} : \mathbb{C} \longrightarrow U_1 \subset S^2$.



Write $z = x + iy$. We can calculate

$$\phi_1^{-1}(z) = \left(\frac{2x}{|z|^2+1} , \frac{2y}{|z|^2+1} , \frac{|z|^2-1}{|z|^2+1} \right).$$

Note that if we have a sequence z_j with $|z_j| \to \infty$ then $\phi_1^{-1}(z_j) \longrightarrow (0,0,1) = NP$.

It follows that S^2 is homeomorphic to $\mathbb{C}_\infty$ and hence that $\mathbb{C}_\infty$ has a Riemann surface structure.

The idea that motivates the construction of Riemann surfaces is that we can do complex analysis with them.

We should know for example what il

means for a complex valued function on a Riemann surface to be "holomorphic" and "holomorphic functions" should behave the way we expect them to.