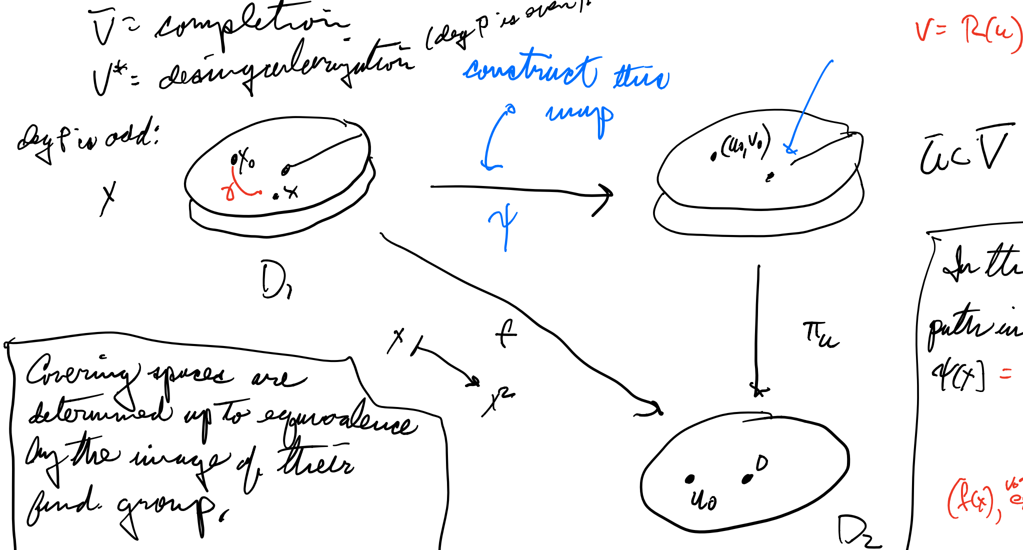
$V = \{(z,w) : w^2 = P(z)\}$

$\bar{V}$ = completion

V^* = desingularization (deg P is even).

Near (∞,∞): $\bar{V} = \{(u,v) : v^2 = R(u)\}$.

deg P is odd:



Covering spaces are determined up to equivalence by the image of their fund. group.

In the language of path integrals

$\psi(x) = (f(x), v_0 h(\gamma))$

$= \left(f(x), \exp \int_{f(\gamma)} \frac{R'}{R}\, du\right)$

$\left(f(x), v_0 \exp\left(\frac{1}{2}\int_{f(\gamma)} \frac{R'}{R}\, du\right)\right)$

need to check ind. of path.

This is exactly the condition that

Covering spaces are determined the image of the fundamental group. ψ is an

$$f_*\left(\pi_1(D_1 - \{0\})\right) = (\pi_u)_*\left(\pi_1(\bar{U} - \infty)\right).$$

loops in D_2 which go around 0 an even # of times

We can describe this in the language of $h(\gamma)$.

Construct ψ by path integration determine when the answer $\int_\gamma$ depends on γ

When is $h(\gamma_0) = h(\gamma_1)$ or $h(\gamma_0 \cdot \gamma_1^{-1}) = 1$?

Alternate (simpler construction). ②

To build $\psi(x) = (x^2, \sqrt{R(x^2)})$

$R(x) = x^n \cdot G(x)$ G holomorphic $G(0) \neq 0$.

$R(x^2) = x^{2n} \cdot G(x^2)$

Square root is given by $x^n \cdot \sqrt{G(x^2)}$. exists and is hol. in a nbd of 0. since $x \mapsto G(x^2)$ is not 0 at 0.

$\psi(x) = (x^2, x^n \sqrt{G(x^2)})$

Let's modify our language based on our new and more sophisticated view of hyper-elliptic surfaces.

Let V^* denote the compact Riemann surface obtained by "resolving any singularities of $\bar{V}$.

V^* comes together with a map to $\mathbb{C}_\infty \times \mathbb{C}_\infty$. $f: V \to \mathbb{C}_\infty \times \mathbb{C}_\infty$

Def. A map from a Riemann surface into $\mathbb{C}_\infty \times \mathbb{C}_\infty$ is holomorphic if for each coordinate function $\pi_j : \mathbb{C}_\infty \times \mathbb{C}_\infty \to \mathbb{C}_\infty$ $\pi_j \circ f$ is holomorphic. (3)

The inclusion of $\bar{V} \to \mathbb{C}_\infty \times \mathbb{C}_\infty$ induces a map from $V \to \bar{V} \to \mathbb{C}_\infty \times \mathbb{C}_\infty$.

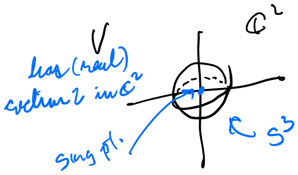
Observation $V^* \to \mathbb{C}_\infty \times \mathbb{C}_\infty$ is holomorphic $p_1, p_2 \mapsto (\infty, \infty)$ in every case

(but not a holomorphic embedding).

Remark. In constructing V^* we have also constructed two meromorphic functions on V^*, π_z and π_w.

How do we distinguish isolated singular points of $\bar{V}$?

There is an elegant topological invariant.



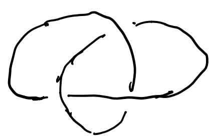
$V \cap S^3$ has real codim 2 in S^3

Construct a small sphere around the singular point. This is S^3.

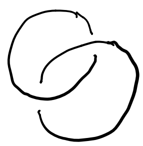
The intersection of V with this sphere is a 1-manifold in S^3 i.e. a knot or a braid. Remove a pt. from S^3 and draw S^3 as $\mathbb{R}^3$.

This braid is a topological invariant of the singular point.

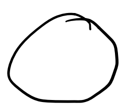
The point (∞, ∞) is a singular point of this holomorphic map.



trefoil knot $\deg P = 3$ $(2,3)$ torus knot.



two linked circles $\deg P$ is even



unknot $\deg P$ is 1

Proposition. Any construction which produces an atlas on the <u>topological</u> surface V^* so that the inclusion of

V^* into $\mathbb{C}^2$ is holomorphic produces a surface <u>conformally equivalent</u> to V^*. (Proved this in connection with polygonal subsets of $\mathbb{R}^3$.)

Proof. Any V' is the alternate Riem. surface structure on V^*. We get a homeomorphism $h: V^* \to V'$ which is holomorphic away from a finite set of points. By removable sing., h is holomorphic everywhere. Similarly h^{-1} is hol. everywhere.

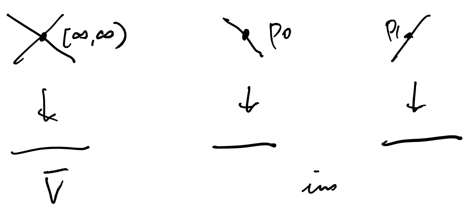
(I think of these alg. singularities as being analogous to cone pts.)

If P has simple zeros then each root of P is valence 2 point of π_z.

If $\deg P$ is odd then ∞ is also a valence 2 point for π_z. If $\deg P$ is even then p_0, p_1 are regular (valence 1) points for π_z.

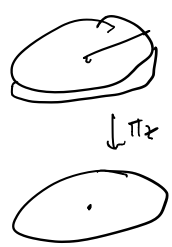
Recall that the valence $v_f(p)$ is the local mult. for points near $f(p)$.

Both are +1.



π_z

If $\deg P$ is odd then π_z is locally 2-1 near (∞,∞).



π_z

locally 2-1 near (∞,∞).

Hyper-elliptic surface is determined up to hol. equiv. by the set of images of the branch points in $\mathbb{CP}^1$.

V^*

Write V_P for the completed, desingularized Riemann surface coming from $w^2 = P(z)$.

Summarizing earlier discussion.

genus of V_P is

$\frac{\deg(P)-1}{2}$ $\deg(P)$ odd

$\frac{\deg(P)}{2} - 1$ $\deg(P)$ even