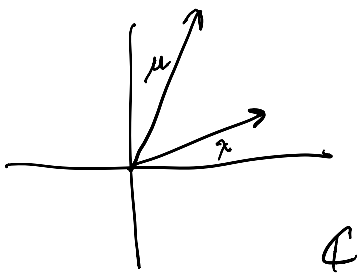
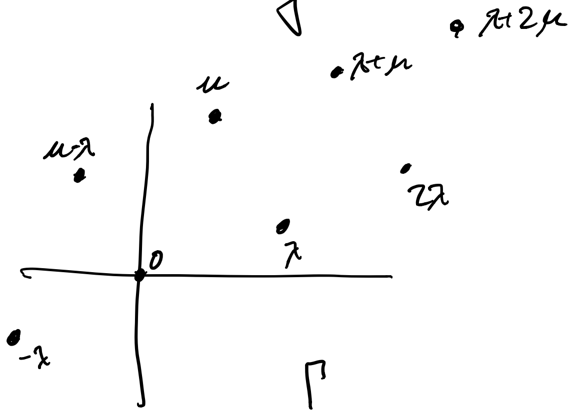
Example sheet 1 has been posted.

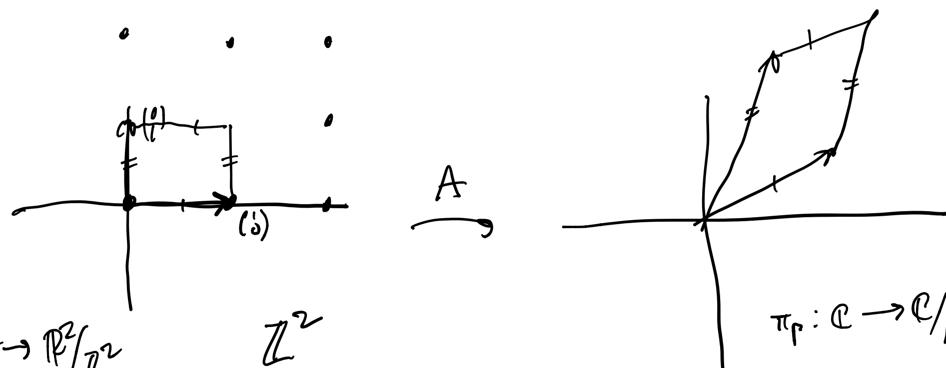
At the end of the last class we were putting a Riemann surface structure on $\mathbb{C}/\Gamma$ where

$\Gamma = \{m\lambda + n\mu : m, n \in \mathbb{Z}\}$

Let me add a little commentary to the previous discussion



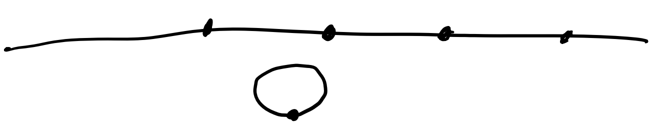




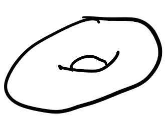
$\pi_{\mathbb{Z}} : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ $\quad \mathbb{Z}^2$ $\quad \pi_\Gamma : \mathbb{C} \to \mathbb{C}/\Gamma$.

$$\frac{\mathbb{R}^2}{\mathbb{Z}^2} \cong \left(\frac{\mathbb{R}}{\mathbb{Z}}\right) \times \left(\frac{\mathbb{R}}{\mathbb{Z}}\right) \cong S^1 \times S^1 = T^2$$

$\mathbb{C}/\Gamma$ is homeomorphic to T^2.

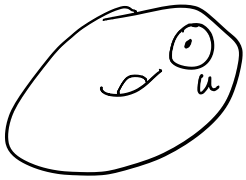


Fact: $\pi_{\mathbb{Z}}$ is a covering map. This implies that π_Γ is a covering map.

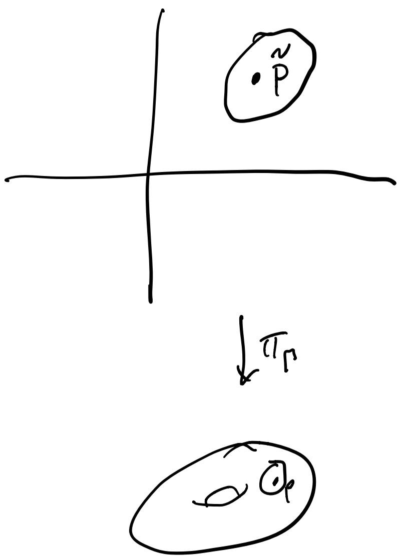


We want to define an atlas of charts on $\mathbb{C}/\Gamma$.

(We want this atlas to be compatible with the existing complex structure on $\mathbb{C}$.



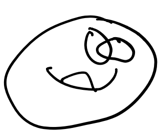
Given a point p in $\mathbb{C}/\Gamma$ we want a nbd. U of p. Want to choose a lift $\tilde{p}$ of p and a nbd $U_{\tilde{p}}$ of $\tilde{p}$ which is disjoint from all of its translates by Γ.

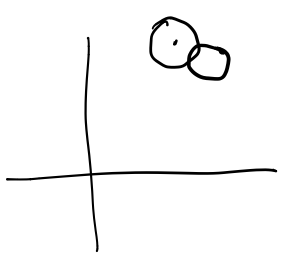


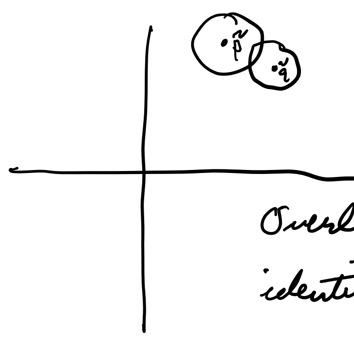
We can do this because π_Γ is a covering map.

Now $U_{\tilde{p}}$ is a subset of $\mathbb{C}$ so we define a chart $\phi_p = (\pi_\Gamma | U_{\tilde{p}})^{-1}$.

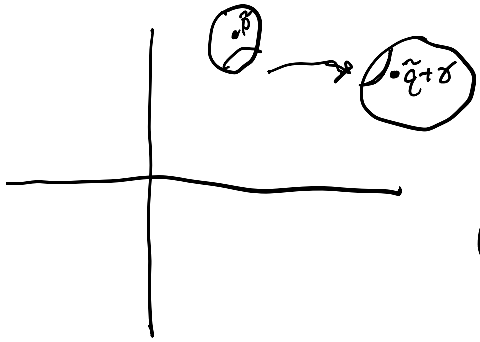
What do the overlap functions look like?







Overlap is the identity



Overlap is a translation by $\gamma \in \Gamma$.

(This is clearly holomorphic.)

Definition. We say that two Riemann surfaces R, S are holomorphically equivalent if there there is a holomorphic map $f: R \to S$ with a holomorphic inverse.

(Of course f is conformal so we sometimes say that R and S are conformally equivalent.)

We have constructed a family of Riemann surfaces all of which are topologically equivalent to tori and hence all topologically equivalent to each other.

We can ask whether they are conformally equivalent? The answer is no.

We have constructed a large number of conformally distinct surfaces. What is a good way to understand a family of Riemann surfaces?

We will return to this later.

We are going to give another example of a quotient space construction.

$$\mathbb{C}^2 = \{(z,w) : z,w \in \mathbb{C}\}$$

$$W = \mathbb{C}^2 - \{(0,0)\}.$$

Define an equivalence relation $\sim$ on W by

$(z,w) \sim (u,v)$ iff for some $t \in \mathbb{C}$ $(z,w) = (tu,tv)$.

note $t \in \mathbb{C} - \{0\}$.

Remark that these equivalence classes are orbits of a group action where $\mathbb{C}^* = \mathbb{C} - \{0\}$ is given the group structure coming from multiplication and it acts on $\mathbb{C}^2$ by scalar mult. of a vector.

Write the equivalence class of (z,w) as $[z:w]$ homogeneous coordinates (by definition z,w not both 0)

Let $\mathbb{C}P^1$ be the set of equivalence classes

$$\begin{tikzcd} W \arrow[rr, "q"] \arrow[dr, "\alpha"'] & & \mathbb{C}P^1 \arrow[dl, "\beta"] \\ & \mathbb{C}_\infty & \end{tikzcd}$$

$q((z,w)) = [z:w]$

$$\alpha((z,w)) = \frac{z}{w} \quad \text{if } w \neq 0$$
$$= \infty \qquad w = 0.$$

$$\beta([z:w]) = \frac{z}{w} \qquad w \neq 0$$
$$\infty \qquad w = 0.$$

$\mathbb{C} \to S^2$. If we pull back the atlas on S^2 to $\mathbb{CP}^1$ we get $U_1 = \{[z:w] : w \neq 0\}$

$U_2 = \{[z:w] : z \neq 0\}$

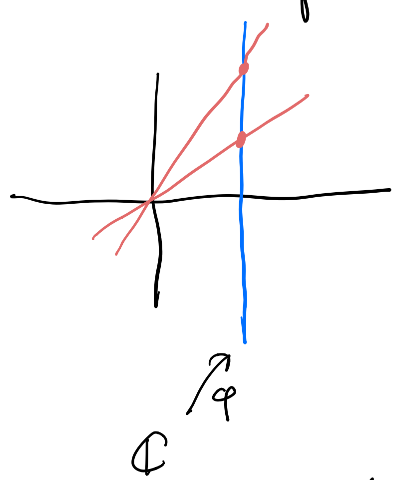
$\varphi_1 : U_1 \to \mathbb{C}$ $\quad \varphi_1([z:w]) = \frac{z}{w}$

$\varphi_2 : U_2 \to \mathbb{C}$ $\quad \varphi_2([z:w]) = \frac{w}{z}$.

This gives $\mathbb{CP}^1$ a Riemann surface structure.

Could add additional charts $\varphi_*([z:w]) = \frac{az+bw}{cz+dw} \in \mathbb{C}$. defined where $cz+dw \neq 0$, $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$.

Geometric interpretation:



Identify a line with its inverse slope.

Map a copy of $\mathbb{C}$ to a line which does not go through the origin.

If ℓ is the line through the origin parallel to the image of psi then we get a chart on $\mathbb{CP}^1 - \ell$ by sending ℓ' to z where

This construction of $\mathbb{CP}^1$ $(\cong S^2)$ suggests symmetries of $\mathbb{CP}^1$.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is any 2×2 complex matrix, with $\det \neq 0$ then

the map $\begin{pmatrix} z \\ w \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix}$ preserves the previous

atlas. In terms of the chart φ_1 above

we have $\varphi_1^{-1}(z) = \begin{pmatrix} z \\ 1 \end{pmatrix} \xmapsto{A} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix}$

$$\varphi_1 \circ A \circ \varphi_1^{-1}(z) = \frac{az+b}{cz+d}$$ $\rightsquigarrow$ gives us a linear fractional transformation.