Beardon p.8

Prop. If $f$ is holomorphic and non-constant on a connected domain $U$ then at each $z \in U$ there is some first coefficient so

$$f(z) = f(z_0) + (z-z_0)^N \left( a_N + a_{N+1}(z-z_0) + \dots \right)$$

with $a_N \neq 0$.

Proof. The set where $a_j = 0$ ("$a_j(z)$") is closed since $a_j$ varies continuously with $z$ so the set where all $a_j$ vanish is closed.

On the other hand the set where all $a_j$ vanish is open since by analyticity if all coefficients vanish at a point then then $f$ is const. in a nbd. of that point.

So the set where all $a_n$ vanish is open and closed. By connectivity of $U$ it is empty or all of $U$.

If it were all of $U$ then $f$ would be constant (using connectivity a second time). Since the set where all $a_n$ vanish is not all of $U$ it is empty.

**Corollary.** Given a domain $U$ and a non-constant holomorphic function $f$ on $U$ and $w_0 \in \mathbb{C}$, the set of $z_0$ s.t. $f(z_0) = w_0$ is isolated.

**Proof.** Say $f(z_0) = w_0$ then near $z_0$

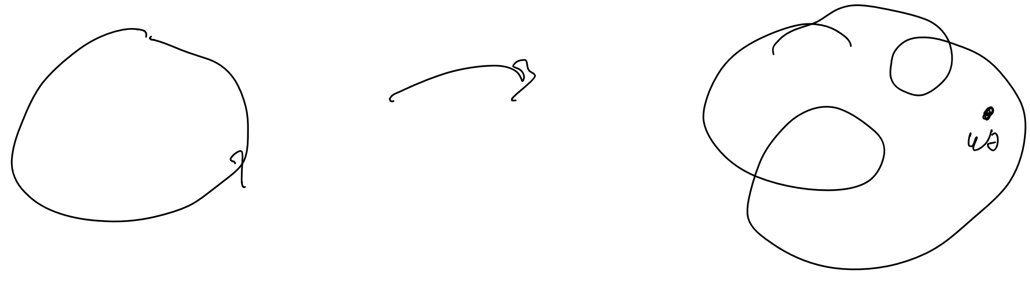
$$f(z) = f(z_0) + (z - z_0)^n g(z) \quad \text{with } g(z_0) \neq 0$$

so $f(z) - w_0 = (z - z_0)^n g(z)$. If $z \neq z_0$ is close to $z$ then this difference is non-zero by the cont. of $g$.

"Argument principle" in Degree theory

**Local degree.** Let $D$ be a disk (say) and $f$ a non-constant holomorphic function on $D$.

Consider a point $w_0 \in \mathbb{C}$.



Assume

What is the winding number of $f(\partial D)$ around $w_0$?

$$\int_{f\circ\gamma} \frac{dz}{z-w_0} = \int_\gamma f^*\left(\frac{dz}{z-w_0}\right)$$

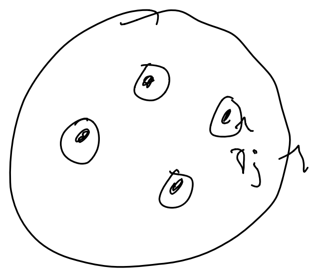
$$= \int_\gamma f^*\left(\frac{1}{z-w_0}\right) f^*(dz).$$

$d f^*(z)$

$$= \int_\gamma \frac{1}{f(z)-w_0} f'(z)\cdot dz$$

$$= \int_\gamma \frac{f'(z)}{f(z)-w_0}\, dz.$$

The integrand is holomorphic away from points $z$ where $f(z)=w_0$.




As we have seen these points are isolated. Let $z_j$ be one of them.

Let $\gamma_j$ be a small circle around $z_j$.

$$f(z) = w_0 + (z - z_j)^n g(z) \quad \text{with } g(z_j) \neq 0$$

$$\frac{1}{2\pi i} \int_{\gamma_j} \frac{f'(z)}{f(z) - w_0}\, dz = \frac{1}{2\pi i} \frac{n(z-z_j)^{n-1} g(z) + (z-z_j)^n g'(z)}{(z-z_j)^n g(z)}$$

$$= \frac{1}{2\pi i} \int_{\gamma_j} n \frac{dz}{z} + \int_{\gamma_j} \frac{g'(z)}{g(z)}\, dz$$

$$= n.$$

Let us call this $n$ the <u>valence</u> of $f$ at $z_j$. We interpret it as the multiplicity of $z_j$ as a solution of

$f(z) = w_0$. Notice that (unlike in topology) it is always positive.

Gauss Thm. gives:

Thm. Winding # of $f(\partial D)$ around $w_0$ is the number of solutions of $f(z) = w_0$ counted with multiplicity. (Sum of 7 contributions is zero when we use the clockwise orientations around small circles.

---

Example of a residue calculation:

If $f = \sum_{j=-N}^{\infty} a_n z^n$ has a pole at $0$ then the residue of $f(z)dz$ at $0$ is $a_{-1}$. Path integral can be calculated by summing residues.

---

Recall the Residue Theorem generalizes this analysis. Observe that the residue for us is something attached to a 1-form $f(z)dz$ at a pole of $f$.

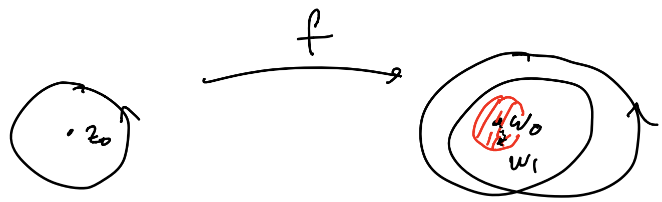
(See this above where form. for change of variable of a 1-form introduces a factor of $f'(z)$.

Corollary: Open mapping theorem.

Theorem. If $f : U \to \mathbb{C}$ is holomorphic but not constant then $f$ takes open sets to open sets.

Proof. Let $z_0 \in U$. Write $f(z_0) = w_0$. Choose a disk $D_0$ around $z_0$ so that $z \in D_0 - \{z_0\} \Rightarrow f(z) \neq w_0$. In particular $f(\partial D_0)$ is disjoint from $w_0$.

$\mathrm{wind}(f(\partial D), w_0)$ = valence of $f$ at $w_0 \geq 1$.




$p(t)$ is a path with $p(0) = w_0$, $p(1) = w_1$.

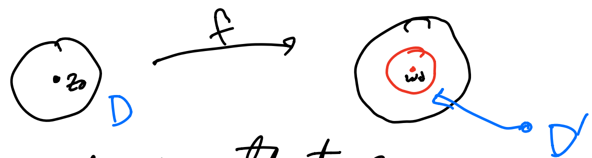
By compacts of $\partial D$ there is a disk $D'$ around $w_0$ which is disjoint from $f(\partial D)$. For any $w_1$ in $D'$ we have the winding number of $f(\partial D)$ around $w_1$ is equal to the winding number around $w_0$ since the winding number is continuous (integral representation) and

$\mathrm{wind}(f\gamma, p(t)) = \int_\gamma \frac{f'(z)\,dz}{z - p(t)}$ cont. fcn. of $t$, integral valued.

integer valued (topological interpretation).
Since this winding number counts the solutions of $f(z) = w_1$ we conclude that $w_1$ is in the image $f(\Omega)$.

Inverse function theorem. Say $f'(z_0) \neq 0$ then $f$ has a local inverse.

Proof. Consider the disks as constructed




previously we see that

$$f(z-z_0) = f(z_0) + (z-z_0)^n g(z) \quad g(z_0) \neq 0 \quad \text{with } n \geq 1$$

Now $f' = 0$ if $n > 0$ so $n = 1$

We conclude that every point in $D'$ has 1 preimage. Now consider the following modification of the winding number integrand

$$\frac{1}{2\pi i} \int_\gamma f(z) \frac{f'(z)}{f(z)-w} dz \quad \text{vs.} \quad \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)-w} dz.$$

Claim that the left hand integral gives

$$\sum z_j \cdot (\text{valence of } z).$$

$z_j : f(z) = w_0$ — mult. of $z_j$ as a soln. of $f(z)=w_0$

Pullback of $\frac{z\,dz}{z-w}$

Cauchy's thm. gives $\frac{1}{2\pi i}\int_{f\gamma} \frac{z\,dz}{z-w} = w_0 \cdot$ [illegible]

Since there is only one solution and the multiplicity is 1 the integral is point $z$ which maps to $w$. Holomorphicity of the inverse follows by differentiating (wrt $w$) under the integral sign.

$$w \mapsto \int_\gamma f(z)\cdot \frac{f'(z)}{f(z)-w}\,dz$$

Definition. A holomorphic $f$ map from $U^{\subset\mathbb{C}} \to V^{\subset\mathbb{C}}$ is <u>conformal</u> if $f'(z) \neq 0$ for $z \in U$.

Cor. A bijective conformal map has a conformal inverse.

Proof. Since $f$ is bijective it has an inverse. The inverse map is locally holomorphic hence holomorphic.

Introduce Riemann surfaces at this pt.

Manifold formalism.

Smooth n-dim manifold / Riemann surface

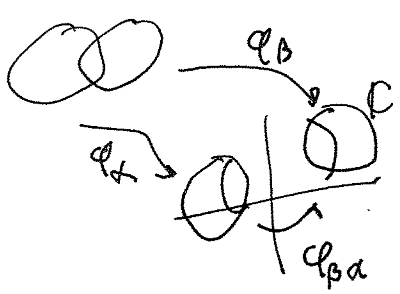


Let $X$ be a topological space (~~say~~ assume Hausdorff)
Let $\mathcal{A}$ be an index set (often finite)
Let $\{U_\alpha\}$ be a collection of open sets in $X$ so that

$X = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$.

An atlas for $X$ is given by a collection of charts
$\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}$ where $\varphi_\alpha$ is a homeomorphism and $V_\alpha$ is an open set in $\mathbb{C}$,

and a collection of transition functions:
$\varphi_{\alpha\beta}$ defined when $U_\alpha \cap U_\beta \neq \phi$ so that



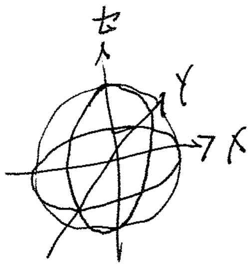

$\varphi_{\beta\alpha} \circ \varphi_\alpha = \varphi_\beta$ on $U_\alpha \cap U_\beta$.

and $\varphi_{\beta\alpha}$ is a conformal map. (automorphism?)

Remarks: If we only assume that $f$ that $f$ is smooth and not conformal then we get a surface.
If we replace $\mathbb{C}$ by $\mathbb{R}^n$ and require trans. maps to be smooth we get a smooth $n$ manifold.
If we replace $\mathbb{C}$ by $\mathbb{C}^n$ and require trans. maps to be holomorphic we get a complex $n$ manifold.

Riemann surface structure on $S^2$?

Let $S^2 = \{(x,y,t) : x^2+y^2+t^2 = 1\}$.




Use coordin $(x,y,t)$.

Let $NP = (0,0,1)$, $SP = (0,0,-1)$.

$U_1 = S^2 - NP$, $U_2 = S^2 - SP$.

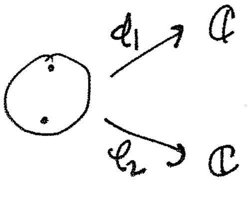
$V_1 = \mathbb{C}$. Identify $\mathbb{C}$ with the $x,y$ plane in $\mathbb{R}^3$.
coordinates by mapping $x+iy$ to $(x,y,0)$

For any pt. $P = (x,y,t) \neq NP$ draw the line from $NP$ to $P$, intersect with the $t=0$ plane and identify $(x,y,0)$ with $x+iy \in \mathbb{C}$.

Formula for the line $s \mapsto (1-s)(0,0,1) + s(x,y,t)$.
Intersection parameter is $s$ s.t. $(1-s)+st = 0$, $s = \frac{1}{1-t}$.
Intersection point is $\left(\frac{x}{1-t}, \frac{y}{1-t}, 0\right)$. Identify this with $\frac{x+iy}{1-t} \in \mathbb{C}$. So $\varphi_1(x,y,t) = \frac{x+iy}{1-t}$.

To define $\varphi_2$ we do the same construction starting with SP. In this case we use a different identification of $t=0$ with $\mathbb{C}$, send $(x,y,0)$ to $x-iy$ and get
$\varphi_2(x,y,t) = \frac{x-iy}{1+t}$ defined on $U_2 = S^2 - SP$ $V_2 = \mathbb{C}$.

(If we had not done this we would have gotten an orientation reversing angle preserving map.)




(Best to think of two distinct copies of $\mathbb{C}$)

We calculate that for $p \overset{(x,y,t)}{=} \in S^2 - \{NP, SP\} = U_1 \cap U_2$

$$\phi_1(p) \cdot \phi_2(p) = \frac{x+iy}{1-t} \cdot \frac{x-iy}{1+t} = \frac{x^2+y^2}{1-t^2} = 1. \quad \left(\begin{smallmatrix}\text{complex}\\ \text{mult.}\end{smallmatrix}\right)$$

since $x^2+y^2+t^2=1$, $x^2+y^2=1-t^2$.

Solve for $\phi_{21}$ using the fact that it satisfies $\phi_{21} \circ \phi_1 = \phi_2$.

~~Set~~. Now $\phi_{21}(\phi_1(p)) = \phi_2(p) = \dfrac{1}{\phi_1(p)}$.

So ~~$\phi_{21}$~~ setting $\phi_{21}(z) = \frac{1}{z}$ we have $\phi_{21} \circ \phi_1 = \phi_2$.

Since $z \mapsto \frac{1}{z}$ is conformal on $\mathbb{C} - \{0\}$ we have a conformal (or holomorphic) atlas.