Example. When we talk about the residue of a complex function f at an isolated singularity z_0 this notion <u>depends</u> on our choice of coordinate.

(Residue as value of a_{-1} in the Laurent expansion of f at z_0.) Laurent expansion is different in different coordinate systems.

Better to think of the residue of the hol. 1-form $f(z)dz$. (Residue as $\frac{1}{2\pi i}\int_\gamma f(z)dz$ where γ is a small anticlockwise loop around z_0 with $\text{wind}(\gamma, z_0) = 1$.)

Every connected Riemann surface can be written as S/Γ where S $= \tilde{R}$ is simply connected and Γ $=$ deck group is acting by holomorphic automorphisms.

We claim that every simply connected Riemann surface is S^2, $\mathbb{C}$ or the disk. We will now describe these automorphism groups.

Automorphisms of $\mathbb{C}_\infty$.

Prop. Every automorphism of $\mathbb{CP}^1$ has the form $z \mapsto \frac{az+b}{cz+d}$.

Proof. Say $f: \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a holomorphic automorphism. f is in particular a meromorphic function so $f(z) = \frac{P(z)}{Q(z)}$ by our previous theorem.

Now $f(z) = w$ has only one solution

$$\frac{P(z)}{Q(z)} = w.$$

May assume that P, Q have distinct roots.

Roots of P map to 0. Thus just one root

$P(z) = c(z-z_0)^n$. $\quad Q(z_0) \neq 0$.

The valence of $\frac{P}{Q}$ at z_0 is n. Since $\frac{P}{Q}$ is an automorphism $V_f(z_0) = n = 1$.

So P is linear.

$$c(z-z_0)^n \left(a_0 + a_1(z-z_0) + \dots\right)$$

$$= c \cdot a_0 (z-z_0)^n + \dots$$

We continue with the disks.

Without loss of generality we can consider the unit disk in $\mathbb{C}$ which we write as Δ.

Schwartz's lemma. Suppose $f: \Delta \to \Delta$ is holomorphic and that $f(0)=0$. Then either

(1) $|f(z)| < |z|$ for every non-zero z in Δ or

(2) $f(z) = e^{i\theta} z$ for some real constant θ.

In addition we have $|f'(0)| \le 1$. If equality holds we are in case (1) above otherwise we are in case 2.

Proof. $f(z) = a_1 z + a_2 z^2 + \dots$
$= z(a_1 + a_2 z + \dots)$
$= z \cdot g(z)$ g holomorphic.

For $r<1$ we can apply the maximum principle to g on the disk $D_r = \{|z| \le r\}$ and obtain

$$|g(z)| \le \sup_{|w|=r} |g(w)| < \frac{1}{r} \qquad (*)$$

since on D_r:

$$1 > |f(z)| = |z \cdot g(z)| = |z| \cdot |g(z)|$$
$$= r \cdot |g(z)|$$

Since max occurs on ∂D_r by the maximum principle.

(Note that there are two versions of the maximum principle. Here we are using the version which says that a cont. fcn. on the closed disk which is holomorphic

in the interior takes its maximum on the boundary. Below we use a refined version which implies that if a holomorphic function on the open disk achieves its maximum then it is constant.)

Letting $r \to 1$ in equation * we get $|g(z)| \leq 1$ in Δ. Note in particular that $|g(0)| = |f'(0)| \leq 1$.

If $|g| = 1$ at some point of the open disk Δ then by the second version of the maximum principle g is constant $g(z) = c$. Thus $|c| = 1$ and $c = e^{ir}$.

g is constant by the refined maximum principal and $g(z) = e^{i\theta} \cdot z$ so (2) holds. Otherwise $|g| < 1$ and (1) holds.

To prove the last two statements note that g is defined and holomorphic on Δ and since $f(z) = z \cdot g(z)$ we have $f'(0) = g(0)$.

If $|f'(0)| = 1$ then $|g|$ achieves its maximum in Δ and we are in case 2. If $|f'(0)| < 1$ then $|g(0)| < 1$ then $|g(z)|$ cannot be 1 at any $z \in \Delta$ so we are in case 1.

Theorem. The elements of $\mathrm{Aut}(\Delta)$ are precisely the Möbius transformations of the form

$$f(z) = \frac{az + \bar{c}}{cz + \bar{a}} \quad \text{with } |a|^2 - |c|^2 = 1.$$

Proof. Elements of this form form a group.