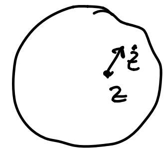
Defined a metric $\|(z,\dot z)\|_{hyp} = \delta(z)\cdot|\dot z|$ =



in x,y coord. $|\dot z|^2 = \delta^2(z)\cdot(dx^2+dy^2)$

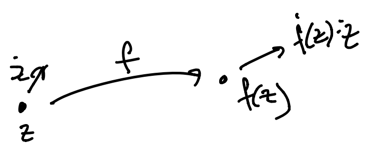
or $ds^2 = \delta^2(z)\,(dx^2+dy^2)$.

$$= \frac{4}{(1-\sqrt{x^2+y^2})^2}(dx^2+dy^2).$$

$$\delta(z) = \frac{2}{1-|z|^2}.$$

If we compare with Euclidean distances

$ds^2 = dx^2+dy^2$ we are just rescaling at each point.

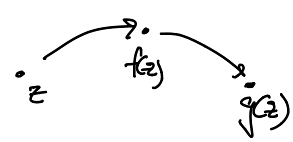


Write $\|f'(z)\|_{hyp}$ for $\frac{|f'(z)\cdot\dot z|}{|\dot z|}$

Check: $\|f'(z)\|_{hyp} = \frac{\delta(f(z))}{\delta(z)}\cdot|f'(z)|$. Note that this is independent of $\dot z$. $(\dot z\neq 0)$.

Check.

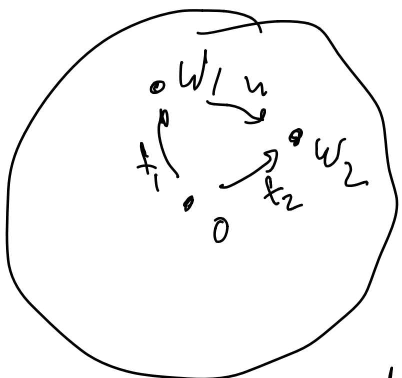
$\|f'(z)\|_{Hyp}\cdot\|g'(f(z))\|_{Hyp} = \|g\circ f\|_{Hyp}$.



$$|f'(z)|\cdot\frac{|\delta(f(z))|}{|\delta(z)|}\cdot|g'(f(z))|\cdot\frac{|\delta(fg(z))|}{|\delta(f(z))|} = |(g\circ f)'|\cdot\frac{|\delta(fg(z))|}{|\delta(z)|}.$$

Thm. (Schwarz–Pick). Let h be any holomorphic map $h : \Delta \to \Delta$. Then h does not increase the hyperbolic distance.

Proof. Say $h(w_1) = w_2$



Say $f_1(0) = w_1$, $f_2(0) = w_2$. Now $f_2^{-1} \circ h \circ f_1$ takes 0 to 0 and takes Δ to Δ. We can apply the standard Schwarz lemma to get

$$\left|(f_2^{-1} \circ h \circ f_1)'\right|(0) \leq 1.$$

Measuring the effect on the hyperbolic metric gives

$$\|(f_2')\|_{hyp} \cdot \|h'\|_{hyp} \cdot \|f_1'\|_{hyp} \leq 1$$

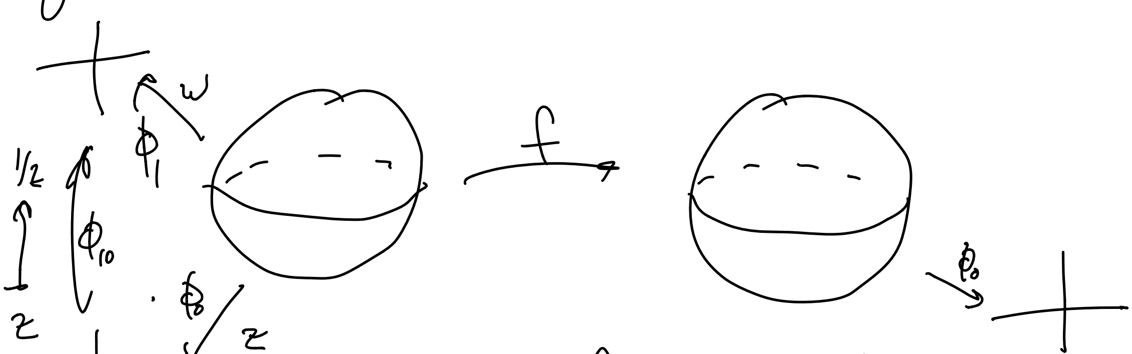
so $\|h'\|_{h} \leq 1$.

Automorphisms of the complex plane.

Theorem. A holomorphic function $f: \mathbb{C} \to \mathbb{C}$ is in $\mathrm{Aut}(\mathbb{C})$ if and only if $f(z) = az + b$ for some constants $a \neq 0$ and b

Proof. Say that $f: \mathbb{C} \to \mathbb{C}$ is holomorphic and injective. We can view f as a hol. map $\mathbb{C} \xrightarrow{f} \mathbb{C}_\infty$.

Is it possible to extend f a holomorphic function whose domain is $\mathbb{C}_\infty$?



$w = \frac{1}{z}$

Does $f \circ \phi_1^{-1}$ have a meromorphic extension?

Now $f(z) = f(\phi_0^{-1}(z)) = f((\phi_{10} \circ \phi_0)^{-1}(w)) = f(\phi_{10}(\phi_0^{-1}(z))) = f(\ )$.

In our notation $f(z) = f(\phi_0^{-1}(z))$.

So $f(\phi_1^{-1}(w)) = f((\phi_\infty \phi_0)^{-1}(w)) = f(\frac{1}{w})$.

Set $F(w) = f(\frac{1}{w})$.

F is defined on $\{0 < |w| < \infty\}$ so F has an <u>isolated</u> singular point at 0.

What kind of isolated singularity is this?

If so then $f \circ \phi_1^{-1}$ would be meromorphic.

Write $F(w) = f(1/w)$.

$f \circ \phi_1^{-1}(w) = f(\frac{1}{w})$ would have a pole at $w=0$.

Now F is defined in $\{0 < |w| < 1\}$ and 0 is an <u>isolated</u> singular point.

It is either removable, a pole or an essential singularity. Which?

If it were essential then the image of the upper hemisphere $\{|z|>1\} = \{|w|<1\}$ would be dense in $\mathbb{C}$. But since f and hence F is injective the image of the upper hemisphere contains no points in the image of the lower hemisphere so 0 is not essential. In particular

∞ is a removable singularity
or a pole for f and
f does extend a holomorphic map
from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$ (taking ∞ to a finite pt. or ∞).

So $f(z) = \dfrac{P(z)}{Q(z)}$. But we also know that $f(\mathbb{C}) = \mathbb{C}$

so there are no poles and Q must have deg 0

ie $f(z) = P(z)$ and P is an injective map so P

must have deg 1.