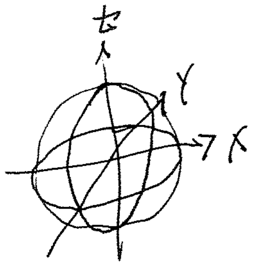


Riemann surface structure on S^2.

Let $S^2 = \{(X,Y,t) : X^2+Y^2+t^2 = 1\}$.

Let $NP = (0,0,1)$, $SP = (0,0,-1)$.

Use coordinates (X,Y,t).

$U_1 = S^2 - NP$, $U_2 = S^2 - SP$.

$V_1 = \mathbb{C}$. Identify $\mathbb{C}$ with the X,Y plane in $\mathbb{R}^3$. coordinates by mapping $X+iY$ to $(X,Y,0)$

For any pt. $P = (X,Y,t) \neq NP$ draw the line from NP to P, intersect with the $t=0$ plane and identify $(X,Y,0)$ with $X+iY \in \mathbb{C}$.

Formula for the line $s \mapsto (1-s)(0,0,1) + s(X,Y,t)$.

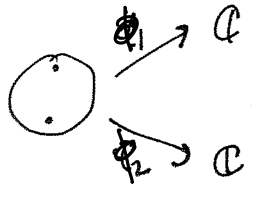
Intersection parameter is s s.t. $(1-s)+st=0$, $s = \frac{1}{1-t}$.

Intersection point is $\left(\frac{X}{1-t}, \frac{Y}{1-t}, 0\right)$. Identify this with $\frac{X+iY}{1-t} \in \mathbb{C}$. So $\Phi_1(X,Y,t) = \frac{X+iY}{1-t}$.

To define Φ_2 we do the same construction starting with SP. In this case we use a different identification of $t=0$ with $\mathbb{C}$, send $(X,Y,0)$ to $X-iY$ and get

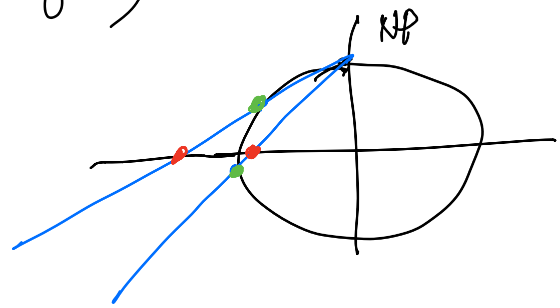
$\Phi_2(X,Y,t) = \frac{X-iY}{1+t}$ defined on $U_2 = S^2 - SP$ $V_2 = \mathbb{C}$.

(If we had not done this we would have gotten an orientation reversing angle preserving map.)

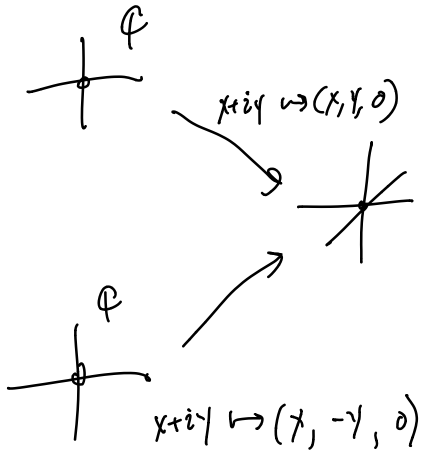


(Best to think of two distinct copies of $\mathbb{C}$)

(In the definition of the surface structure we can allow multiple copies of $\mathbb{C}$.)



$(x,y) \hookrightarrow (x,y,t)$



Neither of these maps is "holomorphic". The question of Holomorphicity first arises with respect to the overlap functions.

We calculate that for $p \overset{(x,y,t)}{=} \in S^2 - \{NP, SP\} = U_1 \cap U_2$

$$\phi_1(p) \cdot \phi_2(p) = \frac{x+iy}{1-t} \cdot \frac{x-iy}{1+t} = \frac{x^2+y^2}{1-t^2} = 1. \quad \left(\begin{matrix}\text{complex}\\ \text{mult.}\end{matrix}\right)$$

↳ product in $\mathbb{C}$.

since $x^2+y^2+t^2=1$, $x^2+y^2 = 1-t^2$.

complex # ↓

Now $\phi_{21}(\phi_1(p)) = \phi_2(p) = \dfrac{1}{\phi_1(p)}$. ↪ inverse of complex #.

Solve for ϕ_{21} using the fact that it satisfies $\phi_{21} \circ \phi_1 = \phi_2$.

Write z for $\phi_1(p)$ then:

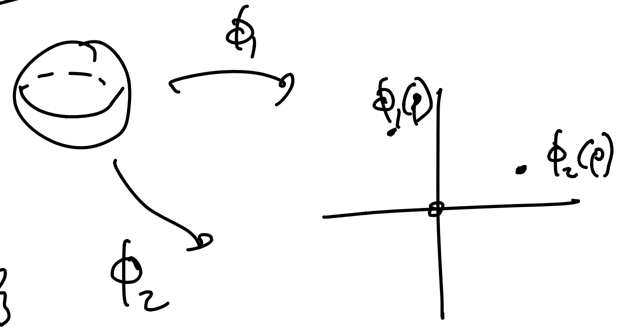
So setting $\phi_{21}(z) = \frac{1}{z}$ we have $\phi_{21} \circ \phi_1 = \phi_2$.

Since $z \mapsto \frac{1}{z}$ is conformal on $\mathbb{C} - \{0\}$ we have a <u>conformal (or holomorphic) atlas</u>.

Analogously $\phi_{12}(z) = \frac{1}{z}$.

$W_1 = \mathbb{C} - \{0\}$

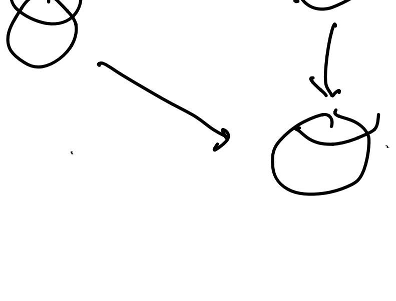
$W_2 = \mathbb{C} - \{0\}$.



$U_1 \cap U_2 = S^2 - \{NP, SP\}$

$W_1 \cap W_2 = \mathbb{C} - \{0\}$

$\phi_{21} : W_1 \cap W_2 \to W_1 \cap W_2$



atlas, S is a Riemann surface.

For our second construction, we take the compactification $\mathbb{C}_\infty$ of $\mathbb{C}$ by adjoining the single point ∞ to $\mathbb{C}$ (see Section 2.10). An explicit homeomorphism of $\mathbb{C}_\infty$ onto S can be constructed as follows. For each z $(= x+iy)$ in $\mathbb{C}$, we project z linearly towards or away from ζ_1 until it meets S at a point $(\neq \zeta_1)$ which we denote by $p(z)$. A computation shows that

$$p(z) = \left(\frac{2x}{|z|^2+1} , \frac{2y}{|z|^2+1} , \frac{|z|^2-1}{|z|^2+1} \right) \tag{3.4.1}$$

Recall that the 1pt. compactification of a locally compact space is obtained by adding a point ∞ and declaring the nbds. of ∞ to be complements of compact sets.

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and we define $p(\infty)$ to be ζ_1: see Figure 3.4.1. Obviously, p is a homeomorphism of $\mathbb{C}$ onto $S - \{\zeta_1\}$. This means that compact subsets of $\mathbb{C}$ correspond to compact subsets of $S - \{\zeta_1\}$ so p is actually a homeomorphism of $\mathbb{C}_\infty$ onto S.

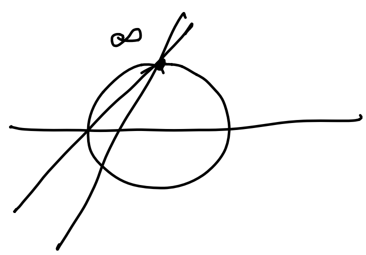
A computation shows that $\phi_1 = p^{-1}$ so using p^{-1} to transfer the atlas on S to an atlas on $\mathbb{C}_\infty$ we arrive at the atlas

$$\sigma_1(z) = \phi_1 p(z) = z \text{ on } \mathbb{C} ,$$
$$\sigma_2(z) = \phi_2 p(z) = 1/z \text{ on } \mathbb{C}_\infty - \{0\}$$

on $\mathbb{C}_\infty$. We call $\mathbb{C}_\infty$ the <u>extended complex plane</u>: it is a Riemann surface homeomorphic to S and being compact, cannot be extended any further (Theorem 3.1.3).

So p extends to a homeomorphism.

$$\phi_1(x,y,t) = \frac{x+iy}{1-t}$$



$\sigma_1(x,y)$

Consider line

$$s \mapsto (1-s)(0,0,1) + s(x,y,t).$$

$(sx, sy, (1-s)+st)$ Set $t=0$

$(sx, sy, (1-s))$ has norm 1

$$s^2x^2+s^2y^2+(1-s)^2 = 1$$

$$s^2(x^2+y^2)+1-2s+s^2=1$$

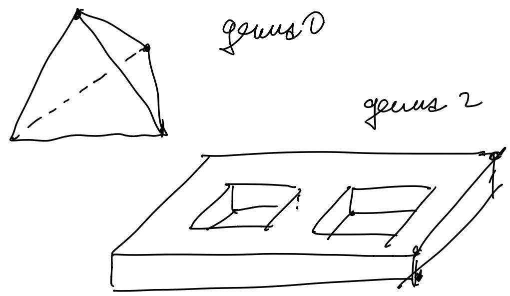
$$s^2(x^2+y^2+1)-2s=0$$

Solve for s → $s(x^2+y^2+1)-2=0$

$$s = \frac{2}{|z|^2+1} = \frac{2}{z\bar{z}+1}$$

$p(z) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$

p induces a homeo. from $\mathbb{C} \to S^2 - \{NP\}$.
Set $p(\infty) = NP$.
Compact sets of $\mathbb{C}$ correspond to compact sets of $S^2 - \{NP\}$. So $p : \mathbb{C}_\infty \to S^2$ is a homeo.



Polyhedral surface.

Polyhedron in $\mathbb{R}^3$ is a topological surface consisting of faces, edges and vertices.

We will show that a polyhedron determines a Riemann surface. This construction gives a large number of examples (unlike previous constructions) including examples of Riemann surface structures on oriented surfaces of every genus. These constructions each have a finite # of parameters that can be adjusted.

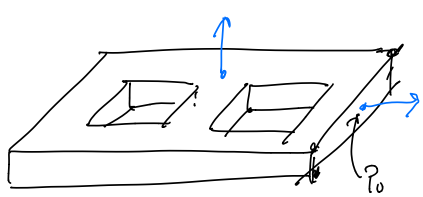
Step 1. We will start by constructing charts at interior points of faces.

Our surface comes with an outward pointing unit normal.

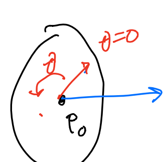
$x+iy \mapsto (x, y, 0)$

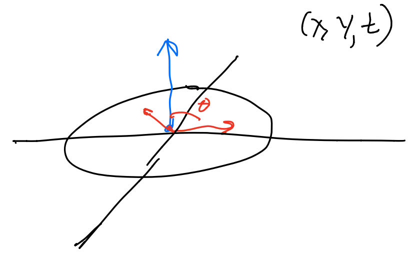
$\mathbb{C} \to \mathbb{R}^3$

(x, y, t)



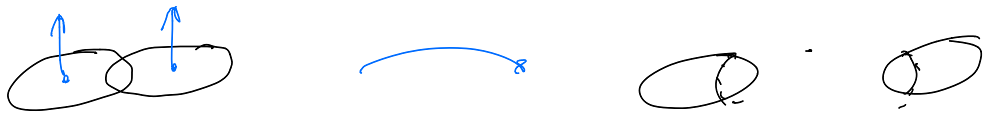
Polyhedral surface.



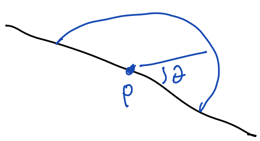


Construct polar coordinates in a nbd. of p_0. $0 \le r \le \delta_0$ $0 \le \theta \le 2\pi$.

If two points are in the same face then nbhds. can overlap and transition functions have the form $z \mapsto z+c$



Step 2, Coordinates on edges



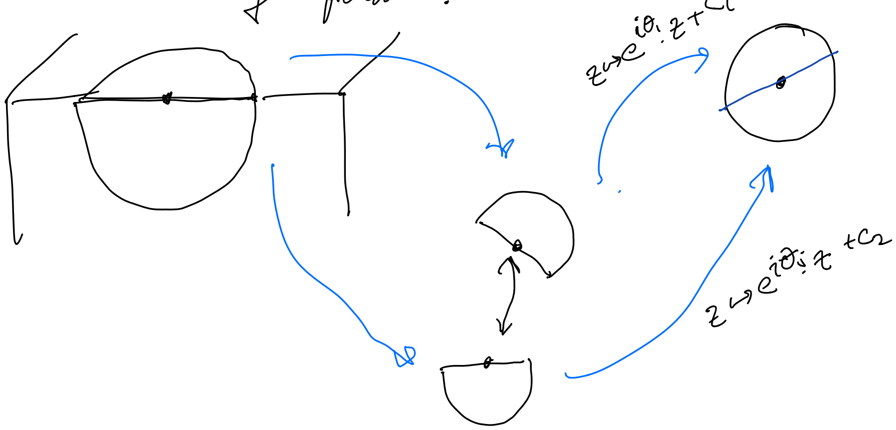
At a point p on the boundary of a face f_i we can construct a half-disk coordinate.

$0 \le r \le s_p$

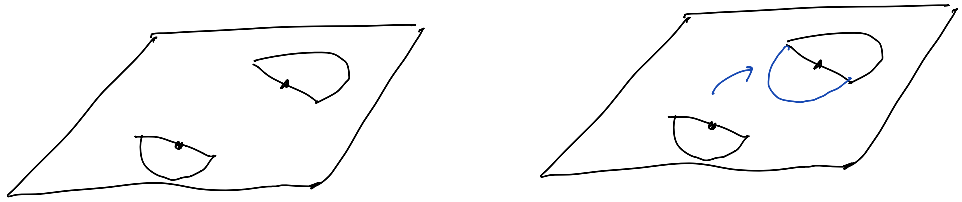
$0 \le \theta \le \pi$.

If we "put together" a pair of these maps we get a disk coordinate.

Edge points.



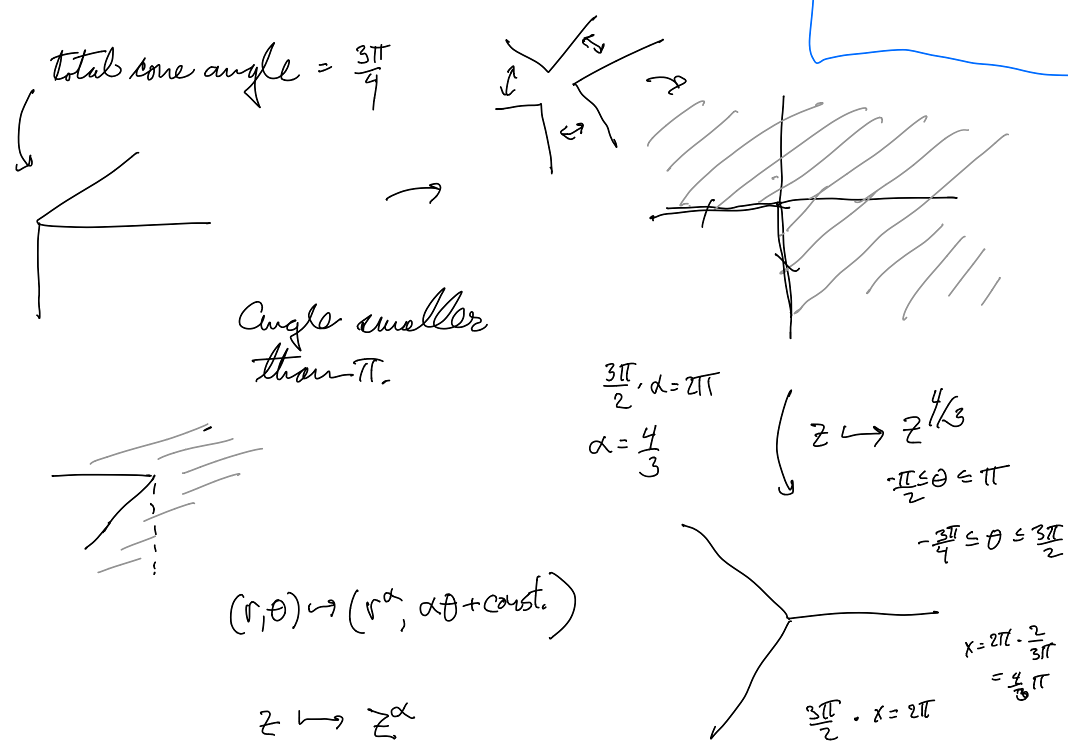
Decide which half-disk becomes the upper half and which the lower half.



At this point we have an atlas for P-vertices where all of the transition maps have the form $\phi_{jk}(z) \mapsto e^{i\theta_{jk}} z + c_{jk}$. This is a holomorphic atlas of a special form and we will see this again.

(r,θ) coordinates
$0 \le r \le \delta_p$
$\theta_0 \le \theta \le \theta_1$

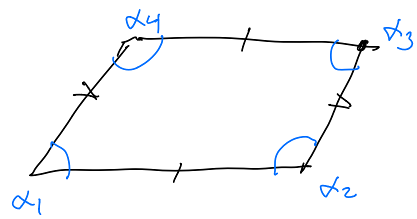
Step 3. Coordinates at vertices.



Adding in charts of this form give us transition functions of the form $z \mapsto \lambda z^{\beta} + c$ where z^{β} is a branch of the power function.

ended here

Remark. Construction can be applied to a collection of polygons in $\mathbb{R}^2$ with identification of sides which need not be realizable in $\mathbb{R}^3$



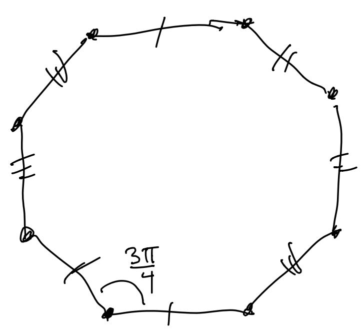
Parallelogram. Identify opposite sides.

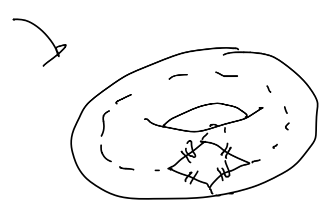
In this case the "vertex" has cone angle

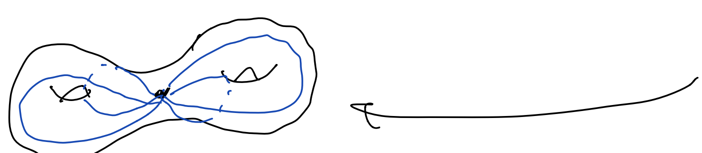
$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \pi.$

Orientation reversing maps of sides (To create orientation preserving charts.)

Octagon:







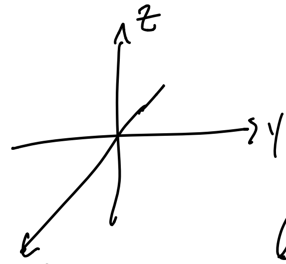
In this case all of the vertices get identified to a single point and the cone angle at that point is $8 \cdot \frac{3\pi}{4} = 6\pi.$

There are two types of pathological behavior that can occur:

① a priori a surface need not be Hausdorff

② a surface need not have a countable base for its topology.

Here is a simple example of a Riemann surface which is neither Hausdorff nor 2nd countable:



Let $P_a = \{(x,y,z) : z=a\}$. Give $\mathbb{R}^3$ the top. of a disjoint union of surfaces P_α.

Form the quotient space by identifying (x,y,z) and (x,y,z') if $y>0$.

(For a more sophisticated example see Beardon.)

The first property *is* compatible with a Riemann surface structure and we will assume that our Riemann surfaces are Hausdorff.

It is a theorem (which will be a corollary of something that we prove) that a connected Hausdorff Riemann surface is 2nd countable.

Example: $\mathbb{RP}^1$

and we define $p(\infty)$ to be ζ_1: see Figure 3.4.1. Obviously, p is a homeomorphism of $\mathbb{C}$ onto $S - \{\zeta_1\}$. This means that compact subsets of $\mathbb{C}$ correspond to compact subsets of $S - \{\zeta_1\}$ so p is actually a homeomorphism of $\mathbb{C}_\infty$ onto S.

A computation shows that $\phi_1 = p^{-1}$ so using p^{-1} to transfer the atlas on S to an atlas on $\mathbb{C}_\infty$ we arrive at the atlas

$$\sigma_1(z) = \phi_1 p(z) = z \text{ on } \mathbb{C} ,$$
$$\sigma_2(z) = \phi_2 p(z) = 1/z \text{ on } \mathbb{C}_\infty - \{0\}$$

on $\mathbb{C}_\infty$. We call $\mathbb{C}_\infty$ the extended complex plane: it is a Riemann surface homeomorphic to S and being compact, cannot be extended any further (Theorem 3.1.3).

For our third model we begin with the space

$$W = \mathbb{C} \times \mathbb{C} - \{(0,0)\}$$
$$= \{(z,w) : z,w \in \mathbb{C}, |z|^2 + |w|^2 \neq 0\}$$

with the subspace topology derived from the product topology on $\mathbb{C} \times \mathbb{C}$. Next, we say that (z,w) and (u,v) are equivalent if there is some complex number t (necessarily non-zero) with $(z,w) = (tu,tv)$. This is an equivalence relation on W: the equivalence class containing (z,w) is

$$[z,w] = \{(tz,tw) : t \in \mathbb{C}, t \neq 0\}.$$

The quotient map $q : (z,w) \mapsto [z,w]$ maps W onto the space $\mathbb{P}$ of equivalence classes and we give $\mathbb{P}$ the quotient topology induced by $q : W \to \mathbb{P}$. We call $\mathbb{P}$ complex projective space.

There are natural maps $\alpha : W \to \mathbb{C}_\infty$ and $\beta : \mathbb{P} \to \mathbb{C}_\infty$ defined by

$$\alpha(z,w) = \beta([z,w]) = \begin{cases} z/w & \text{if } w \neq 0 ; \\ \infty & \text{if } w = 0 \end{cases}$$

so $\alpha = \beta q$. It is easy to see that α is continuous, open and surjective and, as β is 1-1, we see that β is a homeomorphism of $\mathbb{P}$ onto $\mathbb{C}_\infty$ (Theorem 2.7.2): see Figure 3.4.2.

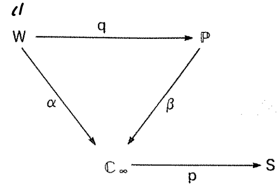


Figure 3.4.2.

Again, we can transfer the atlas on $\mathbb{C}_\infty$ to $\mathbb{P}$ and using β^{-1} this yields $\mathbb{P}$ as a Riemann surface with atlas

$$[z,w] \mapsto z/w \quad \text{on} \quad U_1 = \{[z,w] : w \neq 0\} ,$$
$$[z,w] \mapsto w/z \quad \text{on} \quad U_2 = \{[z,w] : z \neq 0\}.$$

Exercise 3.4

1. Verify (3.4.1) and that $\phi_1 = p^{-1}$.
2. Show that

$$d(z,w) = |p(z) - p(w)| = \frac{2|z-w|}{(1+|z|^2)^{1/2}(1+|w|^2)^{1/2}}$$

is a metric on $\mathbb{C}_\infty$ and that the metric topology is the given topology. We call d the chordal metric on $\mathbb{C}_\infty$ and with this metric, $p : \mathbb{C}_\infty \to S$ is an isometry. Prove that

$$d(1/z,1/w) = d(z,w).$$

3. Verify that $\alpha : W \to \mathbb{C}_\infty$ is open and continuous.

4. Let Q be a plane in $\mathbb{R}^3$ which meets S. Using (3.4.1), show that $p^{-1}(Q \cap S)$ is a circle in $\mathbb{C}$ (if $\zeta_1 \notin Q$) or is $L \cup \{\infty\}$ for some straight line L (if $\zeta_1 \in Q$). For this reason, we usually regard $L \cup \{\infty\}$ as a circle in $\mathbb{C}_\infty$.

5. Writing the elements of W as column vectors, a 2×2 non-singular matrix A acts on W by the rule

$$A : \begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} .$$

$\mathbb{C}^2 = \{(z,w) : z,w \in \mathbb{C}\}$

$W = \mathbb{C}^2 - \{(0,0)\}$.

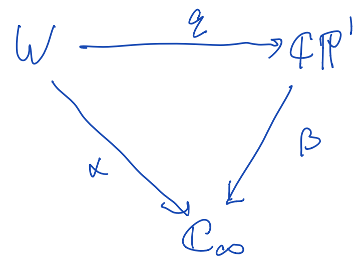
Define an equivalence relation $\sim$ on W by

$(z,w) \sim (u,v)$ iff for some $t \in \mathbb{C}$ $(z,w) = (tu, tv)$.

note $t \in \mathbb{C} - \{0\}$.

Write the equivalence class of (z,w) as $[z:w]$.

Let $\mathbb{CP}^1$ be the set of equivalence classes



$$\alpha((z,w)) = \frac{z}{w} \text{ if } w \neq 0$$
$$= \infty \quad w = 0.$$

$$\beta([z:w]) = \frac{z}{w} \quad w \neq 0$$
$$\infty \quad w = 0.$$

$\mathbb{C} \to S^2$. If we pull back the atlas on S^2

to $\mathbb{CP}^1$ we get $U_1 = \{[z:w] : w \neq 0\}$

$U_2 = \{[z:w] : z \neq 0\}$

$$\varphi_1 : U_1 \to \mathbb{C} \qquad \varphi_1([z:w]) = \frac{z}{w}$$

$$\varphi_2 : U_2 \to \mathbb{C} \qquad \varphi_2([z:w]) = \frac{w}{z}.$$

This gives $\mathbb{CP}^1$ a Riemann surface structure.