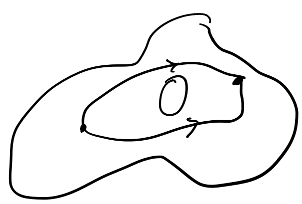
Theorem. Path integration defines a homomorphism from $\pi_1(U, z_0)$ to $\mathbb{C}$ sending γ to $\int_\gamma f(z)dz$.

Call this h_f.

Analysis of the standard argument for constructing the anti-derivative shows:

An anti-derivative for $f(z)dz$ exists if and only if $h_f : \pi_1(U, z_0) \to \mathbb{C}$ is trivial.

Picture:



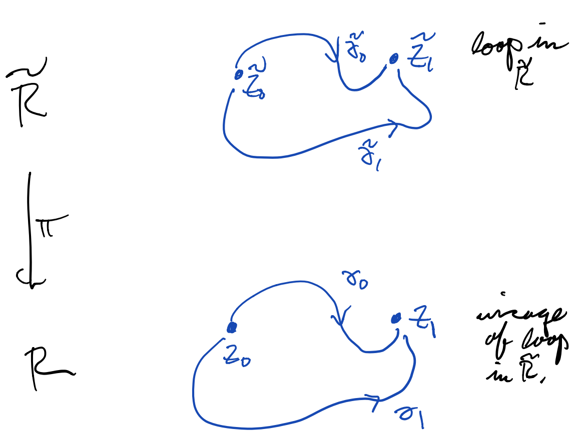
Proposition. Let U be a domain in $\mathbb{C}$. Let f be defined on U. Then we can define an anti-derivative F for f on $\tilde{U}$ where $\tilde{U}$ is the covering space corresponding to the kernel the homomorphism $h_f : \pi_1(U, z_0) \longrightarrow \mathbb{C}$.

Remark. $\tilde{U}$ is the "smallest" cover (fundamental group has largest image) with this property.

(Our anti-derivative construction works to define a holomorphic function.)

Recall how we define the anti-derivative in the simply connected case?

Pick a point z_0. Define $F(z_1) = \int_\gamma f(z)\,dz$ where γ goes from z_0 to z_1.



By definition of the covering space

$\gamma_0 \cdot \gamma_1^{-1} \in \ker$ so

$$\int_{\gamma_0 \cdot \gamma_1^{-1}} f(z)\,dz = 0 \quad \text{so}$$

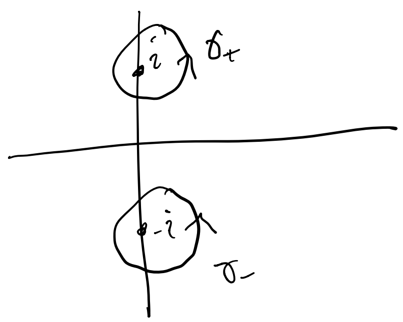
$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

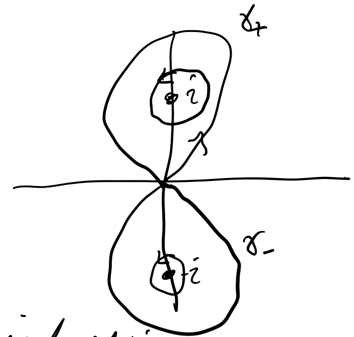
(Note that we are only considering those specific paths downstairs which are images of paths upstairs.)

$\pi_1(U,0) = \langle \gamma^+, \gamma^- \rangle$.

Example. $U = \{\mathbb{C} - \{\pm i\}\}$.

$$f(z) = \frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)}.$$





Calculate residues:

$$\int_{\gamma_+} f(z)\,dz = 2\pi i \operatorname{Res}(f, i)$$

$$\int_{\gamma_-} f(z)\,dz = 2\pi i \operatorname{Res}(f, -i)$$

To find the residue of f at i we expand in terms of $(z-i)$.

$$f = \left(\frac{1}{z-i}\right)\left(\frac{1}{z+i}\right) = \frac{1}{(z-i)}\Big(\underbrace{a_0 + a_1(z-i) + \dots}_{\frac{1}{z+i}}\Big)$$

$$= \frac{a_0}{(z-i)} + a_1 + \dots$$

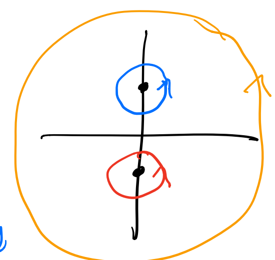
a_0 is the value of $\frac{1}{(z+i)}$ at $z = i$ so $\frac{1}{2i}$

$$\operatorname{Res}(f, i) = \frac{1}{2i}$$

$$\int_{\gamma_+} f \, dz = \frac{2\pi i}{2i} = \pi.$$

$$\int_{\gamma_-} f \, dz = -\pi.$$

Get a covering space which is not a universal cover.

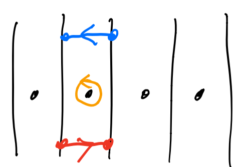


Using facts we know

$$\begin{tikzcd} \mathbb{C} - \{i\mathbb{Z}\} \arrow[dr, "\tan"'] & \tilde{U} \arrow[l, "\sim"', "\text{“arctan”}"] \arrow[d, "\pi"] \\ & \mathbb{C} - \{\pm i\} \end{tikzcd}$$

function we have constructed

Claim: tan lifts to $\tilde{U}$ and gives an inverse



$= \mathbb{C} - \{\pi\mathbb{Z}\}$. not simply connected.

Like log, arctan is not an actual function but it is an actual covering space.

Inverse function is π-periodic (not explained by what we have done)

Why is the anti-derivative well defined on the cover?.

Example: Algebraic varieties.

We will not give a coherent development of these we will discuss some examples.

On the other hand the examples we discuss are historically interesting and perhaps give us some feel for this branch of the field.

Let $P(z)$ be a polynomial $P(z) = c_1 + c_2 z + \dots c_m z^m$.

A hyperelliptic variety has the form

$$R = \{(z, w) : w^2 = P(z)\}.$$

$\boxed{\text{Historical note}}$

Elliptic curves: degree of P is 3 or 4.

Integrals $\int \frac{dz}{\sqrt{P(z)}}$ arise in computing the arclength of an ellipse for example.

Hyper-elliptic curves generalize elliptic curves.