Last time we saw two constructions of conformal structures on the 2-sphere and the construction of families of conformal structures coming from polyhedra.

Question: Is there a coherent way of understanding all Riemann surfaces?

Answer: Yes, in fact, perhaps more than one. In this course we will describe one such approach to tackling this problem.

$$\mathbb{C}^2 = \{(z,w) : z,w \in \mathbb{C}\}$$

$$W = \mathbb{C}^2 - \{(0,0)\}.$$

Define an equivalence relation $\sim$ on W by

$(z,w) \sim (u,v)$ iff for some $t \in \mathbb{C}$ $(z,w) = (tu, tv)$.

note $t \in \mathbb{C} - \{0\}$.

Remark that these equivalence classes are orbits of a group action where $\mathbb{C} - \{0\}$ is given the group structure coming from multiplication.

Write the equivalence class of (z,w) as $[z:w]$ (by definition z,w not both 0)

Let $\mathbb{CP}^1$ be the set of equivalence classes

$$\begin{tikzcd}
W \arrow[rr, "q"] \arrow[dr, "\alpha"'] & & \mathbb{CP}^1 \arrow[dl, "\beta"] \\
& \mathbb{C}_\infty &
\end{tikzcd}$$

$q((z,w)) = [z:w]$

$$\alpha((z,w)) = \frac{z}{w} \text{ if } w \neq 0$$
$$= \infty \qquad w = 0.$$

$$\beta([z:w]) = \frac{z}{w} \qquad w \neq 0$$
$$\infty \qquad w = 0.$$

$\mathbb{C} \to S^2$. If we pull back the atlas on S^2 to $\mathbb{CP}^1$ we get

$$U_1 = \{[z:w] : w \neq 0\}$$

$$U_2 = \{[z:w] : z \neq 0\}$$

$$\varphi_1 : U_1 \to \mathbb{C} \qquad \varphi_1([z:w]) = \frac{z}{w}$$

$$\varphi_2 : U_2 \to \mathbb{C} \qquad \varphi_2([z:w]) = \frac{w}{z}.$$

This gives $\mathbb{CP}^1$ a Riemann surface structure.

Could add additional charts $\varphi_*([z:w]) = \frac{az+bw}{cz+dw} \in \mathbb{C}$. defined where $cz+dw \neq 0$, $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$.

This construction of $\mathbb{CP}^1$ $(=S^2)$ suggests symmetries of $\mathbb{CP}^1$.

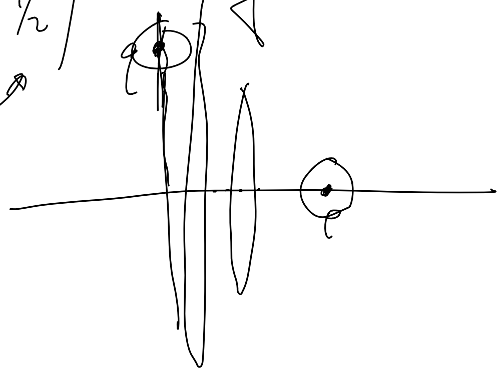
If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is any 2×2 complex matrix, with $\det \neq 0$ then the map $\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix}$ preserves the previous atlas. In terms of the chart φ_1 above

we have $\varphi_1^{-1}(z) = \begin{pmatrix} z \\ 1 \end{pmatrix} \overset{A}{\mapsto} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix}$

$\varphi_1 \circ A \circ \varphi_1^{-1}(z) = \frac{az+b}{cz+d}$ $\rightsquigarrow$ gives us a linear fractional transformation.

Group action which gives a non-Hausdorff manifold as a quotient.

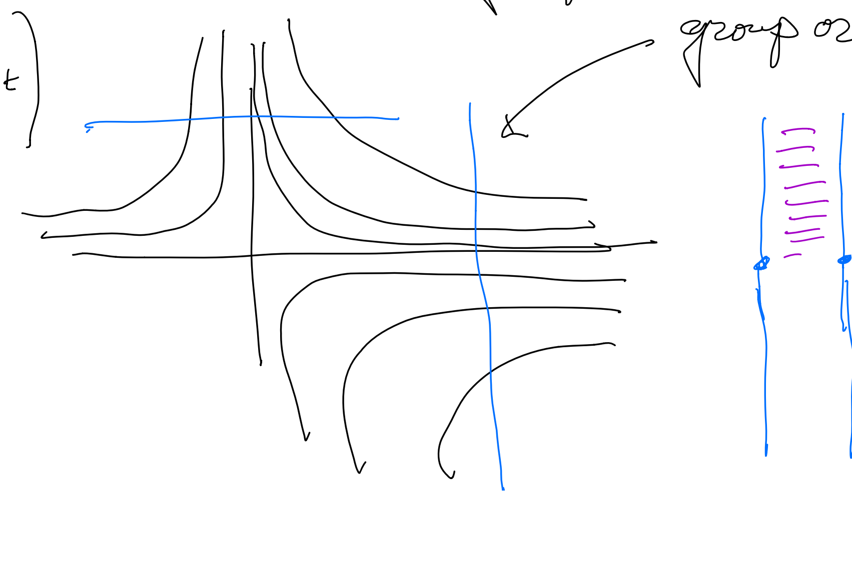
Γ generated by $\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$ acting on $\mathbb{R}^2 - \{(0,0)\}$.



The group action is free which implies that the quotient is a manifold.

In the quotient space p and q do not have disjoint nbds.

To see what the quotient looks like it is useful to extend the group action to $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$



group orbits

Theorem. Let D be a domain in $\mathbb{C}_\infty$ and let Γ be a (discrete) group of Möbius transformations with the property that

(1) $g(D) = D$ for every g in Γ

(2) if $g \in \Gamma$ and $g \neq I$ then the fixed points of g lie outside of D

(3) for each compact set K of D, the set $\{g \in \Gamma : g(K) \cap K \neq \phi\}$ is finite.

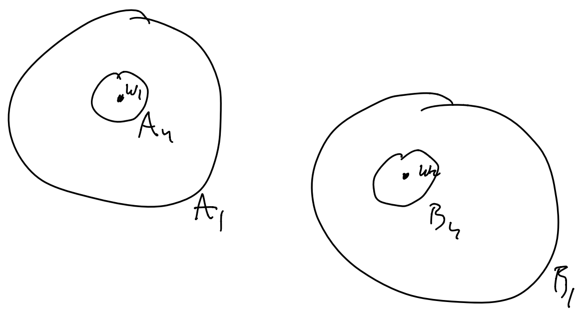
Then the quotient space D/G is Hausdorff and inherits a natural holomorphic atlas.

Proof. Let $q : D \to D/\Gamma$ be the map taking each $z \in D$ to its Γ orbit. Give D/Γ the quotient topology: A set is open in D/Γ if its inverse image is open in D. Thus q is open, continuous and surjective.

Now we want to show that D/Γ is Hausdorff.

Select w_1 and w_2 in D which correspond to distinct points in D/Γ.

Let A_n and B_n be closed balls of radius ε/n around w_1 and w_2 in D where $\varepsilon > 0$ is chosen small enough that $A_n, B_n \subset D$.



Let $K = \bar{A}_1 \cup \bar{B}_1$. For some n we want A_n, B_n to be disjoint but we also want all of their images under Γ to be disjoint.

Suppose that for all n this is not the case thus there are $a_n \in A_n$, $b_n \in B_n$ with $\gamma_n a_n = b_n$. In particular

$$g_n(K) \cap K \supset g_n A_n \cap B_n \neq \emptyset,$$

By hypothesis 3 we have that $\{g_n\}$ is finite. So for an infinite set of indices we have $g_n = g$ for some fixed $g \in \Gamma$.

Since $a_n \in A_n$ we have $\lim a_n = w_1$.

Since $g_n(a_n) \in B_n$ we have $\lim g_n(a_n) = w_2$.

Now consider indices with $g_n = g$ and this gives $a_n \to w_1$ $g(a_n) \to w_2$ so $g(w_1) = w_2$ contrary to our hypothesis.

To construct the atlas for each $z \in D$ we pick a compact nbd K of z. Since for only finitely many g does $g(K) \cap K \neq \emptyset$ we can find a smaller K' which is disjoint from its images. The map $q : \text{int } K' \to D/\Gamma$ is injective.

$$\begin{tikzcd} z \in K' \subset D \arrow[d] \\ D/\Gamma \end{tikzcd}$$

Let $K_z = \text{int } K'$ and $q_z = q|K_z$. q_z is a homeomorphism onto an open subset U_z of D/Γ and we use this map to define an inverse chart from $q_z^{-1} : U_z \to K_z$.

Overlap functions are restrictions of maps $g : D \to D$

and are thus holomorphic.